CHAPTER 11

Social Choice and Political Science

This chapter deals with models of collective choice in which individual agents' preferences are aggregated into collective behavior. The first class of models use some fixed method to aggregate preferences. We assume that collective choices can be observed, but that individual agents' preferences are unobserved. The second class of models are more structured models of voting in political economy and political science. A common idea in political science is that voters' preferences are "Euclidean"; we present the testable implications of this notion. Finally, we consider models of individual voter behavior and work out the corresponding observable implications.

11.1 TESTABLE IMPLICATIONS OF PREFERENCE AGGREGATION FUNCTIONS

The main questions in this section take the following form. Suppose that a group preference (or choice) is observable. Is this group preference consistent with a collection of rational agents whose preferences are aggregated according to some rule? We may, for example, wonder when a group's collective behavior is consistent with majority rule.

There are three ways to interpret the material that we are about to present. First, if we know the aggregation rule that the agents use, we may want to test the hypothesis that a society of agents behave rationally as individuals, when the only observable data come in the form of aggregate preference. Second, when the aggregation rule is unknown, we may want to test the *joint* hypotheses that a group of agents use a certain aggregation rule, and that they each behave rationally as individuals. Finally, a different interpretation of these results is that we might want to characterize all possible "paradoxes" that we might expect from using a given aggregation rule. Condorcet's paradox (a cycle on three alternatives) illustrates the problems that can arise from using majority rule. The results in this section describe all possible paradoxes of this type.

The model is as follows. Let X be a set of possible alternatives. We shall assume that we observe all possible binary comparisons of elements in X; that is, we observe a complete binary relation $\succeq$ on X. This assumption is similar in spirit to our assumption in Chapter 2 that all choice behavior is observable. The set of all complete binary relations will be denoted $\mathcal{C}(X)$. Individual agents' preferences will be strict preferences (linear orders) over X. Denote the set of linear orders over X by $\mathcal{L}(X)$.

We now describe two classical aggregation methods. We fix a finite set N, which we interpret as a set of agents. An *aggregation rule* is a function $f : \mathcal{L}(X)^N \to \mathcal{C}(X)$ mapping *profiles* of strict preferences $(\succeq_i)_{i\in N}$ into a complete binary relation $f((\succeq_i)_{i\in N})$.

The *majority rule* is the aggregation rule f_m defined by

$$(x,y) \in f_m((\succeq_i)_{i\in N}) \text{ iff } |\{i \in N : (x,y) \in \succeq_i\}| \geq \frac{|N|}{2}.$$

Note that, if we let $\succeq = f_m((\succeq_i)_{i\in N})$, then $x \succ y$ iff there is a strict majority of agents who strictly prefer x over y (defining the strict part of $\succeq$ in the usual way).

The *unanimity rule* is the function f_u defined by $(x,y) \in f_u((\succeq_i)_{i\in N})$ if there is $i \in N$ for which $x \succeq_i y$. Note that, if we let $\succeq = f_u((\succeq_i)_{i\in N})$, then $x \succ y$ iff all agents strictly prefer x over y (defining the strict part of $\succeq$ in the usual way).

We can now turn the above definitions into a revealed-preference exercise. Assuming that a complete binary relation $\succeq$ is observed, and given a rule f, we want to know when there is a set of agents N and a preference profile for the agents in N such that $\succeq$ is the image of the profile under f. Say that $\succeq_0 \in \mathcal{C}(X)$ is *majority rationalizable* if there exist a finite N and $(\succeq_i)_{i\in N}$ for which $f_m((\succeq_i)_{i\in N}) = \succeq_0$. Similarly, we will say that $\succeq_0 \in \mathcal{C}(X)$ is *unanimity rationalizable* (or *Pareto rationalizable*) if there exist a finite N and $(\succeq_i)_{i\in N}$ for which $f_u((\succeq_i)_{i\in N}) = \succeq_0$.

Remark 11.1 Note that we allow the freedom of choosing the cardinality of N when we construct a rationalization. Below we turn to problems in which the cardinality of N is fixed.

The first and most important result here is a negative result stating that any $\succeq_0 \in \mathcal{C}(X)$ is majority rationalizable. Therefore, the hypothesis that a social preference arises from application of majority rule to some unknown society is untestable.

McGarvey's Theorem *Any $\succeq_0 \in \mathcal{C}(X)$ is majority rationalizable.*

The negative message in McGarvey's Theorem is not affected by our assumption that all binary comparisons are observable. If one's observations were less complete, then, of course, majority rule would remain non-testable. The conclusion in McGarvey's Theorem carries over to cases in which our observations are poorer than a complete binary relation.

Proof. The proof is constructive. Let $\succeq_0 \in \mathcal{C}(X)$. If $\succeq_0$ is complete indifference, simply let N consist of two agents with exactly opposed preferences. Suppose then that there is at least one pair $x, y \in X$ with $x \succ_0 y$.

First, let us write $X = \{x_1,...,x_m\}$ (this is possible as X is finite). Given any pair $x,y \in X$ for which $x \succ_0 y$, we define two linear-order relations. The first, $\succeq^0_{(x,y)}$ ranks $x \succ^0_{(x,y)} y$, and ranks x and y (strictly) above every element of $X \setminus \{x,y\}$; otherwise, for $x_i, x_k \in X \setminus \{x,y\}$, $x_i \succ^0_{(x,y)} x_k$ iff $i > k$. The second, $\succeq^1_{(x,y)}$, ranks $x \succ^1_{(x,y)} y$, and ranks x and y (strictly) below every element of $X \setminus \{x,y\}$; otherwise, for $x_i, x_k \in X \setminus \{x,y\}$, $x_k \succ^1_{(x,y)} x_i$ iff $i > k$. By defining N to be a set of cardinality $2|\{(x,y) \in X^2 : x \succ_0 y\}|$ and assigning, for each $(x,y) \in \succ_0$, one agent with preference $\succeq^0_{(x,y)}$ and one with preference $\succeq^1_{(x,y)}$, we arrive at $(\succeq_i)_{i \in N}$ for which $f_m((\succeq_i)_{i \in N}) = \succeq_0$.

We now turn to a revealed preference question for the unanimity rule. Our second result states that any quasitransitive relation is unanimity rationalizable.

Theorem 11.2 *A binary relation $\succeq_0 \in \mathcal{C}(X)$ is unanimity rationalizable iff it is quasitransitive.*

Proof. It is easy to see that any unanimity rationalizable relation is quasitransitive. Conversely, let $\succeq_0 \in \mathcal{C}(X)$ be quasitransitive.

We shall show that, for every x,y which are unranked in $\succ_0$, there is a linear order extending $(\succ_0 \cup \{(x,y)\})$. Then let N have a cardinality of

$$2 \times |\{(x,y) \in X^2 : x,y \text{are unranked}\}|.$$

Assign, for each such pair, one agent with a linear order extending $\succ_0 \cup \{(x,y)\}$ and one with a linear order extending $\succ_0 \cup \{(y,x)\}$. Then it is easy to see that $f_u((\succeq_i)_{i \in N}) = \succeq_0$.

Let, then, $x,y \in X$ be unranked according to $(\succ_0 \cup =)$. That is, $x \sim_0 y$ and $x \neq y$. The relation $(\succ_0 \cup =)$ is a partial order. Let $\succeq'$ be the transitive closure of $(\succ_0 \cup = \cup \{(x,y)\})$. We claim that $\succeq'$ is also a partial order. It is clearly antisymmetric and reflexive; it remains to show that it is transitive. But this kind of argument is familiar from Theorem 1.5, so we omit it here. By Theorem 1.4, we know that there is a linear order which extends $\succeq'$.

The previous results deal with groups of unknown size: the size of the group is as unknown as the agents' preferences (see Remark 11.1). Therefore, in constructing a rationalization, one has the freedom of using a group of any size. The construction in each of the proofs illustrates that the groups can be quite large.

It so happens, though, that we frequently know the size of the group, or at least an upper bound. For example one may want to rationalize the behavior of a given committee (such as a faculty meeting, Congress, or the United Nations) when the number of members is known, but not their individual preferences.

It turns out that when we restrict the cardinality of the set of agents, the problem of determining whether a binary relation is majority (or unanimity) rationalizable is much more difficult, and there are far fewer known results.

For an integer n, we say that $\succeq_0$ is *n-unanimity rationalizable* if there exists a set N of cardinality n for which there are $(\succeq_i)_{i \in N}$ such that $f_u((\succeq_i)_{i \in N}) = \succeq_0$.

We could similarly define a related concept for majority rule, but as far as we know there are no results in this direction.

Almost all results for n-unanimity rationalization concern $n = 2$. The next result is due to Dushnik and Miller.

Theorem 11.3 *A binary relation $\succeq_0 \in \mathcal{C}(X)$ is 2-unanimity rationalizable iff $\succeq_0$ is quasitransitive, and there exists a partial order $\succeq^*$ such that $x \sim_0 y$ iff either $x \succeq^* y$ or $y \succeq^* x$.*

Proof. Suppose that $\succeq_0$ is 2-unanimity rationalizable, say by $(\succeq_1, \succeq_2)$. Clearly $\succ_0$ is quasitransitive. Now, define $x \succeq^* y$ if $x \succeq_1 y$ and $y \succeq_2 x$. Note that $\succeq^*$ is a partial order which satisfies the property in the statement of the theorem.

Conversely, suppose that $\succeq^*$ satisfying the property in the statement of the theorem exists. Define a binary relation $\succeq_1$ as follows. Let $x \succeq_1 y$ if either $x \succ_0 y$, or if $x \sim_0 y$ and $x \succeq^* y$. We claim that $\succeq_1$ is a linear order. Completeness is a consequence of the completeness of $\succeq_0$ and the property of $\succeq^*$. To see that it is antisymmetric, suppose that $x \succeq_1 y$ and $y \succeq_1 x$. It cannot be that $x \succ_0 y$, as that would rule out $y \succeq_1 x$. Thus we can assume that $x \sim_0 y$; then $x \succeq_1 y$ implies $x \succeq^* y$, and $y \succeq_1 x$ implies $y \succeq^* x$. It follows that $x = y$, as $\succeq^*$ is a partial order.

Finally, $\succeq_1$ is transitive. Let $x \succeq_1 y$ and $y \succeq_1 z$. There is only something to show when one of these comparisons is due to $\succeq_0$ and the other to $\succeq^*$. These cases are $x \succ_0 y \succeq^* z$ and $x \succeq^* y \succ_0 z$. In the first case, by completeness, if we do not have $x \succeq_1 z$, then we must have $z \succeq_1 x$: (a) If $z \succeq_1 x$ is due to $z \succ_0 x$, we have a contradiction because the transitivity of $\succ_0$ implies that $z \succ_0 y$, which contradicts $y \succeq^* z$; (b) If $z \succeq_1 x$ is due to $z \succeq^* x$, we have a contradiction because the transitivity of $\succeq^*$ implies that $y \succeq^* x$, a contradiction of $x \succ_0 y$. We can derive a similar contradiction in the case $x \succeq^* y \succ_0 z$.

Similarly, we can define the binary relation $\succeq_2$ by $x \succeq_2 y$ if $x \succ_0 y$ or $y \succeq^* x$. Then $\succeq_2$ is a linear order as well.

The linear orders $\succeq_1$ and $\succeq_2$ thus defined provide a rationalization of $\succ_0$. If $x \neq y$, then $x \succ_0 y$ iff $x \succ_1 y$ and $x \succ_2 y$; and $x \sim_0 y$ iff $\succeq_1$ and $\succeq_2$ disagree in how they compare x and y.

In light of Theorem 11.2, Theorem 11.3 gives the property of $\succeq_0$ that, *in addition* to being unanimity rationalizable, it is rationalizable by a group of two agents. This condition, that $\sim_0$ can be "oriented" to form a transitive $\succeq^*$, is difficult to falsify. One would need to check all possible orientations of $\sim_0$. A falsifiable characterization of when such an orientation is possible is provided in the next result.

We first need a simple definition. We will say $x_1, ..., x_k$ is an *odd $\sim_0$ cycle* if k is odd, and for all i, $x_i \sim_0 x_{i+1}$ (as usual, addition is modulo k, and we allow repetitions of vertices). We say the cycle is *triangulated* if there is $i \in \{1, ..., k\}$ for which $x_i \sim_0 x_{i+2}$.

Theorem 11.4 *A binary relation $\succeq_0 \in \mathcal{C}(X)$ is 2-unanimity rationalizable iff it is quasitransitive and every odd $\sim_0$ cycle is triangulated.*

Proof. We will establish necessity of the condition only. Sufficiency, while not conceptually difficult, is tedious. Therefore, let us suppose that $\succeq_1$ and $\succeq_2$ are linear orders which 2-unanimity rationalize $\succeq_0$.

To see that any odd $\sim_0$ cycle is triangulated, suppose, by way of contradiction, that there exists an odd $\sim_0$ cycle $x_1,...,x_k$ which is not triangulated. First, it is the case that for all i, $x_i \neq x_{i+1}$; otherwise since $x_{i-1} \sim_0 x_i = x_{i+1}$, we have $x_{i-1} \sim_0 x_{i+1}$, contradicting the fact that the cycle is not triangulated.

Suppose, without loss of generality, that $x_1 \succ_1 x_2$ and $x_2 \succ_2 x_1$. Now, because $x_1,...,x_k$ is not triangulated, it follows that $x_2 \succ_2 x_3$ and $x_3 \succ_1 x_2$; because, if instead $x_2 \succ_1 x_3$ and $x_3 \succ_2 x_2$, we would have $x_1 \succ_1 x_3$ and $x_3 \succ_2 x_1$, contradicting the fact that $x_1,...,x_k$ is not triangulated. By continuing this argument and using the fact that k is odd, we establish that $x_2 \succ_1 x_1$ and $x_1 \succ_2 x_2$, a contradiction.

These results more or less exhaust the known characterizations of unanimity rationalizable relations for fixed and finite numbers of agents in abstract environments. As far as we know, there are no such results for majority rationalizable relations (or for other simple voting rules). Characterizing such relations should be an important goal for future research. There is reason to believe that such characterizations will be, in general, difficult to come by. A classical result in computer science states that, given a quasitransitive relation, determining whether or not n agents suffice to rationalize that relation by unanimity rule is NP-complete for any $n \geq 3$.

11.1.1 Utilitarian rationalizability

Suppose we have given a family of preference relations $\succeq_1,...,\succeq_n$ on a finite set X. Further, let $\succeq_0$ be an arbitrary transitive relation on X. We will say that $\succeq_0$ is *utilitarian rationalizable* by $\succeq_1,...,\succeq_n$ if there exists, for each i, a utility representation $u_i : X \to \mathbf{R}$ for which $x \succeq_i y \leftrightarrow u_i(x) \geq u_i(y)$ such that

- $x \succ_0 y$ implies $\sum_{i=1}^n u_i(x) > \sum_{i=1}^n u_i(y)$
- $x \sim_0 y$ implies $\sum_{i=1}^n u_i(x) = \sum_{i=1}^n u_i(y)$.[1]

If $\succeq_0$ is complete, these conditions are equivalent to the function $u_0 = \sum_{i=1}^n u_i$ being a utility representation for $\succeq_0$.

Our goal is to characterize utilitarian rationalizable relations $\succeq_0$. That is, we want to test the hypothesis that society ranks alternatives according to a utilitarian criterion, where the ordinal content of individual preference is known, but not the cardinal content. The next result is due to Peter Fishburn.

Theorem 11.5 *A transitive relation $\succeq_0$ is utilitarian rationalizable by $\succeq_1$ $,...,\succeq_n$ iff for all finite disjoint sequences $x_1,...,x_K$ and $y_1,...,y_K$* (i.e. *there is no k,l for which $x_k = y_l$, but possibly allowing repetitions), and for all collections*

[1] See our discussion in 10.3 of utilitarianism in the context of bargaining.

of permutations $\sigma_i : K \to K$ (one for each $i = 1,\dots,n$), such that for all $j = 1,...,K$, $x_j \succeq_0 y_j$ and for all $i = 1,\dots,n$, $y_j \succeq_i x_{\sigma_i(j)}$, it follows that for all $j = 1,...,K$, $x_j \sim_0 y_j$ and for all $i = 1,\dots,n$, $y_j \sim_i x_{\sigma_i(j)}$.

Proof. The relation $\succeq_0$ is utilitarian rationalizable iff there exists, for each i and each $x \in X$ a number $u_i^x \in \mathbf{R}$, for which the following inequalities are satisfied:

I) If $x \succeq_0 y$, then $\sum_i u_i^x \geq \sum_i u_i^y$.
II) If $x \succ_0 y$, then $\sum_i u_i^x > \sum_i u_i^y$.
III) If $x \succeq_i y$, then $u_i^x \geq u_i^y$.
IV) If $x \succ_i y$, then $u_i^x > u_i^y$.

We can easily write this in matrix form. Let B be a matrix with $|X \times N|$ columns, and with one row for each constraint listed above. Rows of type (I) and (II) will be specified by the vector $\mathbf{1}_{N\times\{x\}} - \mathbf{1}_{N\times\{y\}}$, while rows of type (III) and (IV) will be of the form $\mathbf{1}_{(i,x)} - \mathbf{1}_{(i,y)}$. We search for the existence of a real-valued vector $u \in \mathbf{R}^{X\times N}$ with the property that for rows m of type (I) or (III), $B_m \cdot u \geq 0$, and for rows m of type (II) or (IV), $B_m \cdot u > 0$. By Lemma 1.13, the non-existence of such a u is equivalent to the existence of an integer-valued vector $\eta \geq 0$, such that for some row m of either type (II) or (IV), $\eta_m > 0$, and $\eta \cdot B = 0$. Rows are indexed by their associated relation; so a row of type (I) will be written $B_{x\succeq_0 y}$, for example.

It cannot be that only constraints of type (III) or (IV) are associated with $\eta_m > 0$, as we know there always exists a utility representation for a weak order on a finite set. So, let us consider two sequences of length $K = \sum_{\{m:m \text{ is of type (I) or (II)}\}} \eta_m$, say $x_1,...,x_K$ and $y_1,...,y_K$, where for fixed $(x,y) \in X\times X$, $|\{k : x_k = x, y_k = y\}| = \eta_{x\succeq_0 y} + \eta_{x\succ_0 y}$.

Since $\succeq_0$ is transitive, we may assume these two sequences are disjoint (that is, there is no j,k for which $x_j = y_k$), as the rows can be canceled to remove overlapping elements. Similarly, since each $\succeq_i$ is transitive, we can assume that there is no triple x,y,z and $i \in N$ for which the rows corresponding to $x\,R_i\,y$ and $y\,Q_i\,z$ each have positive weight, where R and Q can be either $\succeq$ or $\succ$.

Similarly, we may also assume that for any $i = 1,...,n$, and any x, it cannot be the case that there are y,z for which both a row corresponding to $y \succeq_i x$ or $y \succ_i x$ and a row corresponding to $x \succeq_i z$ or $x \succ_i z$ each have positive weight. This we may assure by simply summing such rows to get a row involving only y and z. We will call this assumption (*).

Now, it follows that since $\eta \cdot B = 0$, and since the $x_1,\dots,x_K$ and $y_1,\dots,y_K$ sequences are disjoint, it must be that rows of type (I) or (II) can only be eliminated by a collection of rows of type (III) or (IV). That is, every instance of x_j contributes a positive term $\mathbf{1}_{(i,x_j)}$, so in order for $\eta \cdot B = 0$, this positive term must be canceled by a negative term. This negative term must come from a constraint of type (III) or type (IV), so there must exist, for agent i, a row of type $y \succeq_i x$ or a row of type $y \succ_i x$ with positive weight. And by our assumption (*), this introduces a positive term on $\mathbf{1}_{(y,i)}$, which must be matched by some y_k.

What we have shown is that there exists, for each agent, a bijection $\sigma_i : K \to K$ such that $y_j \succeq_i x_{\sigma_i(j)}$. And since one of the rows corresponds to type (II) or (IV), $\eta_m > 0$, it follows that either there exists $x_j \succ_0 y_j$, or there exists $i \in 1,\dots,n$ such that $y_j \succ_i x_{\sigma_i(j)}$. This is exactly what is precluded by the statement of the theorem.

11.2 MODELS IN FORMAL POLITICAL SCIENCE

It is interesting and fruitful to analyze formal models in political science from the perspective of revealed preference theory. There is a wealth of data on which to test theories of political competition and voters' behavior. Here we shall discuss two different models, and investigate their empirical content under different assumptions about what can be observed.

We first discuss the standard model of "spatial" preference and investigate circumstances where it lacks testable implications, even when we have rich data (we observe a full preference relation). We then turn to more general models of voter behavior, and consider more limited data.

11.2.1 Refuting Euclidean preferences

Many models in political science are based on voters having a special kind of "spatial" preference. We consider here the testable implications of such models of voter behavior. The idea is that policy positions can be represented as points in some Euclidean space, $\mathbf{R}^d$. We can view a vector $x \in \mathbf{R}^d$ as representing a position in each of d issues. We can then model a voter's behavior using a preference relation on $\mathbf{R}^d$.

The standard benchmark model in political science is that of *Euclidean preferences,* where each agent is endowed with an ideal point, and a vector of policy positions is preferred to another iff it is closer to the voter's ideal point. Formally, for each agent i, let $y_i \in \mathbf{R}^d$ be i's *ideal point*. Then i's preference relation, $\succeq_i$, is defined by $x \succeq_i z$ iff $\|x - y_i\| \leq \|z - y_i\|$, where $\|x\| = (\sum_{i=1}^d x_i^2)^{1/2}$ is the Euclidean norm on $\mathbf{R}^d$ (hence the term Euclidean preferences).

The theory is simple, but when we observe data on voter behavior, we do not observe choices among vectors in $\mathbf{R}^d$. One basic problem is then to identify alternatives with vectors in some Euclidean space, in a way that is consistent with the theory.

Given are a finite set X of alternatives and a finite set of agents N. Agents are endowed with preference relations over X, $\succeq_i$, one for each $i \in N$. A preference profile $(\succeq_i)_{i\in N}$ is *Euclidean rationalizable* if there exist a mapping $\rho : X \to \mathbf{R}^d$, and ideal points $y_i \in \mathbf{R}^d$, so that for all $x, z, \in X$ and all $i \in N$, $x \succeq_i z$ iff $\|\rho(x) - y_i\| \leq \|\rho(z) - y_i\|$. Note that we make no requirement that ρ be one-to-one.

In revealed preference theory, we often try to understand the properties of data that refute a given theory. Here we shall instead ask if the theory is at all refutable, without trying to understand the morphology of refutations. So we ask, when is it the case that all preference profiles are Euclidean rationalizable?

The answer to this question will clearly depend on at least three things: $|X|$, $|N|$, and d. If, for some triple $(|X|, |N|, d)$, all preference profiles are Euclidean rationalizable, then the model has, in some sense, no empirical content. If instead there are profiles that are not Euclidean rationalizable, then the theory may have some empirical bite. The following two results are due to Bogomolnaia and Laslier.

Theorem 11.6 *All profiles $(\succeq_i)_{i\in N}$ of preference relations are Euclidean rationalizable iff $d \geq \min\{|X|-1, |N|\}$.*

Proof. We first demonstrate that if $d \geq \min\{|X|-1, |N|\}$, then the Euclidean model has no empirical content. To this end, first suppose that $d \geq |X|-1$. In particular, let us consider the case $d = |X|-1$; the other cases follow trivially from this. We shall carry out a construction for which we do not actually need to know the profile of preferences $\succeq_i$.

We let the set of policy alternatives be the set in $\mathbf{R}^{d+1}$ given by $\{x \in \mathbf{R}^{d+1} : \sum_{j=1}^{d+1} x_i = 1\}$. This set is isomorphic to $\mathbf{R}^d$. Now, we can write $X = \{1, ..., d+1\}$. Consider the mapping ρ which carries each m to the vector $\mathbf{1}_m$ (as usual, $\mathbf{1}_m$ denotes the unit vector with a 1 in entry m, and zeros in all other entries). The simplex defined by $\Delta(d) = \{x \in \mathbf{R}^{d+1} : x \geq 0 \text{ and } \sum_{j=1}^{d+1} = x\}$ can be used to choose the ideal points, y_i. In particular, for $m, l \in X$, we can consider the set $H(m,l) = \{x \in \Delta(d) : x_m \geq x_l\}$, and the set $H^+(m,l) = \{x \in \Delta(d) : x_m > x_l\}$. If $y \in H(m,l)$, then $\|\mathbf{1}_m - y\| \leq \|\mathbf{1}_l - y\|$, and if $y \in H^+(m,l)$, then $\|\mathbf{1}_m - y\| < \|\mathbf{1}_l - y\|$. So, given a preference $\succeq_i$, we need $y_i \in H(m,l)$ for all $m \succeq_i l$ and $y_i \in H^+(m,l)$ for all $m \succ_i l$. It is easy to see that this can always be done. For example, take any utility representation $u_i : X \to \mathbf{R}$ of $\succeq_i$ and consider $-u_i$. Consider the function $v_i(x) = -u_i(x) + \lambda$, where $\lambda > 0$ is chosen large enough so that $v_i(x) > 0$ for all x. Then renormalize by $\alpha > 0$ so that $\sum_{x\in X} \alpha(-u_i(x) + \lambda) = 1$. Then the vector $y_i \in \mathbf{R}^{d+1}$ defined by letting $y_{il} = \alpha(-u_i(l) + \lambda)$ for each l satisfies the desired property.

On the other hand, suppose that $d \geq |N|$. We can suppose that $d = |N|$. The case where $d > |N|$ will easily follow from the argument given here. For each $i \in N$, let $u_i : X \to \mathbf{R}$ be a utility function representing $\succeq_i$. Consider the point $\rho^*(x) \in \mathbf{R}^d$ whose ith coordinate is given by $\rho^*(x)_i = u_i(x)$. Now, consider the function $F : \mathbf{R}^{d+1} \to \mathbf{R}^d$ given by

$$F_i(\alpha, z) = -\alpha(z \cdot z) + z_i;$$

with $\alpha \in \mathbf{R}$ and $z \in \mathbf{R}^d$.

Note that $\nabla_z F(0,z) = I$ for all z (where $\nabla_z F$ here stands for the Jacobian matrix), and that $F_i(0, \rho^*(x)) = u_i(x)$ for $x \in X$. By the implicit function theorem (for example, see Rudin, 1976), for each $x \in X$, there is a neighborhood U_x of 0 such that for all $\alpha \in U_x$, there is a point $\rho^\alpha(x)$ for which $F_i(\alpha, \rho^\alpha(x)) = u_i(x)$. By choosing $\overline{\alpha}$ such that $\overline{\alpha} \in \bigcap_{x\in X} U_x$ and $\overline{\alpha} > 0$, we have for each x a point $\rho^{\overline{\alpha}}(x)$ for which $F_i(\overline{\alpha}, \rho^{\overline{\alpha}}(x)) = u_i(x)$. Thus, $x \mapsto F_i(\overline{\alpha}, \rho^{\overline{\alpha}}(x))$ represents $\succeq_i$ on X. Finally, we claim that for each $i \in N$, $z \mapsto F_i(\overline{\alpha}, z)$ represents a Euclidean preference. We have

$$F_i(\overline{\alpha}, z) = -\left(\overline{\alpha}(z \cdot z) - z_i\right),$$

which is ordinally equivalent to

$$-\left(z - \frac{\mathbf{1}_i}{2\overline{\alpha}}\right) \cdot \left(z - \frac{\mathbf{1}_i}{2\overline{\alpha}}\right),$$

which represents a Euclidean preference with ideal point $\frac{\mathbf{1}_i}{2\overline{\alpha}}$. Let $\rho(x) = \rho^{\overline{\alpha}}(x)$, and we are done.

To show the converse, for every d, we will consider an environment with $d+1$ individuals and $d+2$ alternatives which is not consistent with the Euclidean model. We will demonstrate a list of preferences which cannot be represented. Let us write out the alternatives as $\{x_0, x_1, ..., x_{d+1}\}$. Individual $i \in N$ will have a preference which ranks x_i strictly above all other alternatives, and ranks the remaining alternatives as indifferent. We will argue by contradiction.

The first point to mention is that, by virtue of the preference profile under consideration, it follows that for all $j \neq k$, $\rho(x_j) \neq \rho(x_k)$. That is, ρ is one-to-one.

Though the following proof will hold for arbitrary d, it helps to establish it first in the special case of $d = 1$.

First, for $d = 1$, it is simple to show that there is no representation of this environment. We have two individuals, and $\rho(x_0), \rho(x_1)$, and $\rho(x_2)$ all lie on a straight line. Since $i = 1$ is indifferent between $\rho(x_0)$ and $\rho(x_2)$, $\rho(x_1)$ must be in between $\rho(x_0)$ and $\rho(x_2)$. And since $i = 2$ is indifferent between $\rho(x_0)$ and $\rho(x_1)$, it follows that $\rho(x_1)$ must be in between $\rho(x_0)$ and $\rho(x_1)$. This is clearly impossible.

The proof for $d \geq 2$ relies on an interesting geometric fact. Consider the function $\psi : \mathbf{R}^d \backslash \{0\} \to \mathbf{R}^d$ given by

$$\psi(x) = \frac{x}{\|x\|^2}.$$

This function, known as an *inversion* function in geometry, has two very interesting properties. First, it satisfies $\psi(\psi(x)) = x$ for all $x \in \mathbf{R}^d \backslash \{0\}$. Second, it maps every sphere containing the origin (on its boundary, and of course excluding the origin) to a hyperplane which does not intersect the origin, and of course, therefore maps every hyperplane not containing the origin to a sphere containing the origin (on its boundary). Every hyperplane which passes through the origin is mapped to itself. Lastly, for any point in the interior of any sphere containing the origin, this point is mapped to the opposite side of the origin from the hyperplane which is the image of this sphere. For more on the interesting geometry behind this function, one should consult Coxeter (1969), Chapter 6.

So, now suppose that $d \geq 2$. It is clear that without loss of generality, we may assume that $\rho(x_0) = 0$.

We know that any $(d+1)$-tuple of elements of $\{0, \rho(x_1), \rho(x_2), ..., \rho(x_{d+1})\}$ containing 0 must lie on some sphere (as there is some agent who is indifferent between them all and 0). As a consequence, any set of $\{\psi(\rho(x_1)), ..., \psi(\rho(x_{d+1}))\} \backslash \{\psi(\rho(x_i))\}$ lies on some hyperplane which does not intersect the origin (the image of i's indifference surface), and having the property that $\psi(\rho(x_i))$ lies on the opposite side of the origin from this hyperplane.

We will study the intersection of all hyperplanes containing $\{\psi(\rho(x_1)), ..., \psi(\rho(x_{d+1}))\}$, namely $\mathcal{H} = \{\sum_{i=1}^{d+1} \lambda_i \psi(\rho(x_i)) : \sum_i \lambda_i = 1\}$.[2] We will also make use of the affine set generated by the set $\{\psi(\rho(x_1)), ..., \psi(\rho(x_{d+1}))\} \backslash \{\psi(\rho(x_i))\}$, call this $\mathcal{H}_i$. $\mathcal{H}_i$ lies in a hyperplane which does not pass through the origin, on the other side of which is $\psi(\rho(x_i))$.

We have two possibilities: either $\mathcal{H} = \mathcal{H}_i$ for some i, or not.

Suppose $\mathcal{H} = \mathcal{H}_i$ for some i. Without loss of generality, assume that $\mathcal{H} = \mathcal{H}_{d+1}$. This implies that on any sphere on which $0, \rho(x_1), ..., \rho(x_d)$ all lie, $\rho(x_{d+1})$ must also lie, so that the preference $\succeq_{d+1}$ cannot be represented. This is a contradiction.

For the second case, we note that we may conclude that $0 \in \mathcal{H}$.[3] In particular, there are λ_i for which $\sum_{i=1}^{d+1} \lambda_i = 1$ such that $0 = \sum_{i=1}^{d+1} \lambda_i \psi(\rho(x_i))$. We claim that each $\lambda_i \leq 0$, which will be a contradiction.

We illustrate the case of $i = d+1$. To see this, remember we said that each $\mathcal{H}_{d+1}$ lies on a hyperplane (the image of the indifference surface for agent $d+1$), the other side of the origin from which is $\psi(\rho(x_{d+1}))$. Let the normal of this hyperplane be p_{d+1}. Formally, we have that there is $\alpha_{d+1} > 0$ such that for all $j = 1, \ldots, d$, $p_{d+1} \cdot \psi(\rho(x_j)) = \alpha_{d+1} > 0$, and $p_{d+1} \cdot \psi(\rho(x_{d+1})) > \alpha_{d+1}$. We know that $\sum_{j=1}^{d+1} \lambda_j \psi(\rho(x_j)) = 0$; consequently, $\sum_{j=1}^{d+1} \lambda_j p_{d+1} \cdot \psi(\rho(x_j)) = 0$. However, $\sum_{j=1}^{d+1} \lambda_j p_{d+1} \cdot \psi(\rho(x_j)) = \alpha_{d+1}(\sum_{j=1}^{d} \lambda_j) + \lambda_{d+1} p_{d+1} \cdot \psi(\rho(x_{d+1}))$. If $\lambda_{d+1} \geq 0$, then we have $\alpha_{d+1}(\sum_{j=1}^{d} \lambda_j) + \lambda_{d+1} p_{d+1} \cdot \psi(\rho(x_{d+1})) \geq \alpha_{d+1} \sum_{j=1}^{d+1} \lambda_j = \alpha_{d+1} > 0$, a contradiction. Consequently, $\lambda_{d+1} \leq 0$.

[2] That all hyperplanes containing $\{\psi(\rho(x_i)), \ldots, \psi(\rho(x_{d+1}))\}$ also contain $\mathcal{H}$ is obvious. That each $w \in \mathcal{H}$ is contained in all such hyperplanes can be proved via Lemma 1.12. That is, for vectors $y_1, \ldots, y_m$, $p \cdot y = 0$ is a consequence of $p \cdot y_i = 0$ for all i iff there is $\lambda \in \mathbf{R}^m$ for which $y = \sum_{i=1}^{m} y_i$. Now w is in every hyperplane containing each $\psi(\rho(x_i))$ if for all $p \in \mathbf{R}^{d+1}$ and $\alpha \in \mathbf{R}$, $p \cdot \psi(\rho(x_i)) - \alpha(1) = 0$ implies $p \cdot w - \alpha(1) = 0$. So apply this result to the vectors $(\psi(\rho(x_i)), -1)$ and $(w, -1)$, so that there is $\lambda \in \mathbf{R}^{d+1}$ for which $\sum_{i=1}^{d+1} \lambda_i = 1$ and $w = \sum_{i=1}^{d+1} \lambda_i \psi(\rho(x_i))$.

[3] To see this, note that by dimensionality of the space, there is $\mu \in \mathbf{R}^{d+1} \backslash \{0\}$ for which $\sum_{i=1}^{d+1} \mu_i \psi(\rho(x_i)) = 0$ (the vectors $\psi(\rho(x_i))$ cannot be linearly independent). We must show that $\sum_{i=1}^{d+1} \mu_i \neq 0$. But, since $\mu \neq 0$, the converse would imply that there exists some i for which $\mu_i \neq 0$ and $-\mu_i = \sum_{j \neq i} \mu_j$. Then $-\mu_i \psi(\rho(x_i)) = \sum_{j \neq i} \mu_j \psi(\rho(x_j))$. Dividing each side by $-\mu_i$ now results in $\psi(\rho(x_i)) \in \mathcal{H}_i$, a contradiction.

Theorem 11.6 says that rationalization by Euclidean preferences may require many dimensions. Our next result deals with a weaker theory: voters have convex preferences over policies, but they do not need to be Euclidean. It turns out that two dimensions suffice in this case.

Formally, we say that a profile $(\succeq_i)_{i\in N}$ is *convex rationalizable* if there exists a mapping $\rho : X \to \mathbf{R}^d$, and a convex preference relation $\succeq_i^*$ on $\mathbf{R}^d$ for all $i \in N$, such that for all $x, y \in X$, $x \succeq_i y$ iff $\rho(x) \succeq_i^* \rho(y)$.

Theorem 11.7 *If $d > 1$, $|X| = 2$, or $|N| = 1$, then any profile $(\succeq_i)_{i\in N}$ of preference relations is convex rationalizable.*

We omit the proof of Theorem 11.7, but the basic idea is to let $\rho : X \to \mathbf{R}^d$ be any mapping carrying each X to a distinct vertex of some regular polytope. Then, for each preference $\succeq_i$, we let $U(x) = \mathrm{conv}\{\rho(y) : y \succeq_i x\}$, where conv denotes the convex hull. These are nested. The preferences $\succeq_i^*$ to be constructed must be such that their upper contour sets at x contain $U(x)$. It is always possible to construct such $\succeq^*$.

11.2.2 Rational voting when policy positions are known

We now turn to the case when policy alternatives are already given by vectors in $\mathbf{R}^d$. A voter has preferences over $\mathbf{R}^d$; we are going to consider convex and Euclidean preferences.

Suppose that we observe the behavior of a single voter. A *voting record V* is a finite collection of pairs, $\{(y^k, n^k)\}_{k=1}^K$, where each $y^k, n^k \in \mathbf{R}^d$ and $y^k \neq n^k$. The interpretation is that out of the pair (y^k, n^k), $y^k \in \mathbf{R}^d$ was chosen (voted "yes") while $n^k \in \mathbf{R}^d$ was not.

The environment is a special case of choice theory, as described in Chapter 2. For each k, y^k is chosen over (or "revealed preferred" to) n^k. The set of "budgets" consist of pairs; and we assume that we observe the entire choice function over those budgets (which happens to be single-valued).

By our results in Chapters 2 and 3, we know that any voting record is weakly rationalizable by some utility function (namely, complete indifference). And a voting record is strongly rationalizable iff its revealed preference pair admits no cycles. In this case, we know the revealed preference pair is given by $y^k \succeq^c n^k$ for all k, and $y^k \succ^c n^k$ for all k; that is, $\succeq^c = (\succ^c \cup =)$, so acyclicity of the order pair is the same as acyclicity of $\succ^c$.[4] The first result highlights a reformulation of acyclicity.

A voting record is *strongly pair rationalizable* if there is a utility function $u : \mathbf{R}^d \to \mathbf{R}$ such that, for all k, $u(y^k) > u(n^k)$. The word "pair" in "strongly pair rationalizable" is meant to remind us that data come in the form of pairwise comparisons.

[4] Of course, we also have $y^k \succeq^c y^k$ for all k.

For any $V' \subseteq V$, we define $Y(V') = \{y^k : (y^k, n^k) \in V'\}$ and $N(V') = \{n^k : (y^k, n^k) \in V'\}$. These are the sets of elements that were chosen from V' and rejected from V' respectively. Finally, $X(V') = Y(V') \cup N(V')$.

Proposition 11.8 *A voting record V is strongly pair rationalizable iff for all nonempty $V' \subseteq V$, $Y(V') \neq N(V')$.*

Proof. Suppose that V is strongly rationalizable, and suppose, toward a contradiction, that there is some $V' \subseteq V$ for which $Y(V') = N(V')$. We will show how to construct a $\succ^c$ cycle. Let $n^{k_1} \in N(V')$ be arbitrary; we know that $y^{k_1} \succ^c n^{k_1}$. Moreover, $y^{k_1} \in N(V')$, so there is $(y^{k_2}, n^{k_2}) \in V'$ for which $n^{k_2} = y^{k_1}$. But then we have $y^{n_2} \succ^c n^{k_2} = y^{k_1} \succ^c n^{k_1}$. By continuing this construction inductively, and since V' is finite, we must eventually construct a cycle.

Conversely, suppose that we have $Y(V') \neq N(V')$ for all nonempty $V' \subseteq V$, but that V is not strongly rationalizable. Then we know there is a cycle

$$y^{k_l} \succ^c n^{k_l} = y^{k_{l-1}} \succ^c n^{k_{l-1}} \ldots n^{k_1} = y^{k_l}.$$

By setting $V' = \{(y^{k_i}, n^{k_i})\}$, we have $Y(V') = N(V')$, a contradiction.

An *extreme point* of a set $X \subseteq \mathbf{R}^d$ is a point $x \in X$ which cannot be written as a convex combination of points in $X \backslash \{x\}$. The set of extreme points of X will be denoted $E(X)$. The following theorem is due to Tasos Kalandrakis (as was Proposition 11.8).

Theorem 11.9 *Let V be a voting record. The following conditions are equivalent:*

I) *For all nonempty $V' \subseteq V$, there is $x \in E(X(V')) \backslash Y(V')$.*
II) *There exists a concave utility function strongly pair rationalizing V.*
III) *There exists a strictly concave utility function strongly pair rationalizing V.*
IV) *There exists a quasiconcave utility function strongly pair rationalizing V.*

The main interest is in condition (I). Notice that it is a stronger condition than acyclicity. In fact, we included Proposition 11.8 to highlight the difference between Condition (I) and acyclicity as formulated in Proposition 11.8. If for all nonempty $V' \subseteq V$ there is $x \in E(X(V')) \backslash Y(V')$, then, of course, it is impossible that $Y(V') = N(V')$. The following is a simple example of a strongly pair rationalizable voting record which is not rationalizable by a concave, or a quasiconcave, utility function.

Example 11.10 *Suppose $X = \mathbf{R}$, and that the voting record consists of $V = \{(0,1), (2,1)\}$. Then this voting record can never be strongly rationalized by a concave utility function. This is obvious, but to see this via Theorem 11.9, note that $E(X(V)) = \{0,2\}$, and $Y(V) = \{0,2\}$, a contradiction to condition* (I).

Hence, in the voting model, concavity imposes testable restrictions over and above the restrictions contained in strong rationalizability by a utility function. Contrast this with Afriat's Theorem of Chapter 3, which says that concavity has no testable implications above rationality alone. The difference arises from the kind of data we have assumed here. In the demand theory environment of Chapter 3, we could never directly compare two arbitrary consumption bundles. In the present environment, such comparisons are not only possible, they are the only types of comparisons allowed.

Proof. That the existence of a concave strict rationalization implies Equation (I) is well known, as at least one minimizer of a concave function always occurs at an extreme point. And the minimizer cannot occur at a point $y^k \in Y(V')$, as the corresponding point $n^k \in N(V')$ is ranked strictly lower.

We will show that Equation (I) implies the existence of a concave utility function strictly rationalizing V. To see this, consider the set X. We will show that for each $x \in X(V)$, there is $u_x \in \mathbf{R}$ and $p_x \in \mathbf{R}^d$ such that for all $(x,y) \in V$, $u_x > u_y$, and for all $x,y \in X(V)$, $u_y \leq u_x + p_x \cdot (y-x)$. By defining $u : \mathbf{R}^d \to \mathbf{R}$ by $u(z) = \min_{x \in X(V)} u_x + p_x \cdot (z-x)$, we will have a rationalization. This is the typical Afriat construction; see Chapter 3.

In fact, we shall set up a system like that in the proof of Afriat's Theorem. Define a matrix B as follows. The matrix has $|X(V)| + d|X(V)|$ columns. First are $|X(V)|$ columns, one labeled with each element of $X(V)$. Then a second collection of $d|X(V)|$ columns, these come in groups of d columns, each group is labeled with an element of $X(V)$.

For each pair $(x,y) \in V$, we have a row with a zero in each entry, with the exception of a 1 in the first column labeled with x and a -1 in the first column devoted to y. For each distinct pair $(x,y) \in X(V) \times X(V)$, we have a row with a zero in each entry, with the exception of 1 in the first column labeled with x, -1 in the first column devoted to y, and in the second set of d columns labeled with x we include the vector $(y-x)$. That is, in the first such column we write $y_1 - x_1$; in the second column $y_2 - x_2$, and so on.

We are looking for a utility function $u : X \to \mathbf{R}$ and vector p_x for each $x \in X$ that solve the system of inequalities we described above. We can view a utility function as a vector $u \in \mathbf{R}^{X(V)}$, and stack the d-dimensional vectors p_x in a way that is congruent with the columns of B. Let P be the vector $(p_x)_{x \in X}$ stacked in a manner congruent with the columns of B. Then we can write the system of linear inequalities as

$$B \cdot \begin{bmatrix} u \\ P \end{bmatrix} \geq 0,$$

where the first V inequalities corresponding to u must be strict.

Now suppose that there is no solution to the system of inequalities. By Lemma 1.12, we conclude that for each $(x,y) \in V$, there exists $\lambda_{(x,y)} \geq 0$ and for each $(x,y) \in X(V) \times X(V)$, there is $\mu_{(x,y)} \geq 0$, where at least one λ is strictly positive, and such that for each $x \in X(V)$,

$$\sum_{y \in X(V):(x,y)\in V} \lambda_{(x,y)} - \sum_{y \in X(V):(y,x)\in V} \lambda_{(y,x)} + \sum_{y \in X(V)} \mu_{(x,y)} - \sum_{y \in X(V)} \mu_{(y,x)} = 0 \quad (11.1)$$

and

$$\sum_{y \in X(V)} \mu_{(x,y)}(y - x) = 0. \quad (11.2)$$

Consider all pairs (x,y) for which either $\lambda_{(x,y)} > 0$ or $\mu_{(x,y)} > 0$. Call this set M, and consider $X(M)$. The set M is nonempty; it is not necessarily a subset of V. Note that Equation (11.1) implies that, for any $x \in X(M)$, there is y with $\lambda_{(z,y)} > 0$ or $\mu_{(z,y)} > 0$ (or both).

Next, consider an extreme point z of the set $X(M)$. There is at least one extreme point because $X(M)$ is finite. As $z \in X(M)$, there is y with $\lambda_{(z,y)} > 0$ or $\mu_{(z,y)} > 0$. Now, we cannot have $\mu_{(z,y)} > 0$ for any y because equation (11.2) would imply that z is not an extreme point of $X(M)$.

Therefore, z is associated with a positive $\lambda_{(z,y)}$, for some y. But this implies, in fact, that $(z,y) \in V$. Define $V' \subseteq V$ by $(x,y) \in V'$ iff $\lambda_{(x,y)} > 0$. We know that all extreme points of $X(M)$ are therefore elements of $Y(V')$. But as a consequence, all extreme points of $X(V')$ are also elements of $Y(V')$, which is a contradiction.

Going back to the model discussed in Section 11.2.1, we can ask when a voting record V is rationalizable by a preference relation that is not only convex, but Euclidean. Data $V = \{(y^k, n^k)\}_{k=1}^K$ are rationalizable by the Euclidean model if there is $y \in \mathbf{R}^d$ such that, for all k, $\|y^k - y\| < \|n^k - y\|$.

The following result provides an answer. It states that whenever a weighted average of the y^k vectors and the n^k vectors coincide, it must be the case that the weighted average of the squared lengths of the y^k vectors is strictly less than the weighted average of squared lengths of the n^k vectors.

Theorem 11.11 *Let $V = \{(y^k, n^k)\}_{k=1}^K$ be a voting record. Then there is a Euclidean preference strongly rationalizing V iff for all $\lambda \in \mathbf{R}^k$ for which $\lambda \geq 0$ and $\sum_{k=1}^K \lambda_k = 1$, if $\sum_k \lambda_k y^k = \sum_k \lambda_k n^k$, then $\sum_k \lambda_k \left(y^k \cdot y^k\right) < \sum_k \lambda_k \left(n^k \cdot n^k\right)$.*

Proof. The proof is based on an application of a lemma related to Lemma 1.14. V is rationalizable by a Euclidean preference iff there exists $b \in \mathbf{R}^d$ such that for all $(y^k, n^k) \in V$,

$$-(y^k - b) \cdot (y^k - b) > -(n^k - b) \cdot (n^k - b).$$

Rewriting this equation, we seek the existence of $b \in \mathbf{R}^d$ such that for all k,

$$b \cdot \left(y^k - n^k\right) > \frac{(y^k \cdot y^k) - (n^k \cdot n^k)}{2}.$$

By introducing a variable $\alpha \in \mathbf{R}$, we see that there exists such a $b \in \mathbf{R}^d$ iff there exists $(b,\alpha) \in \mathbf{R}^{d+1}$ for which

$$b \cdot \left(y^k - n^k\right) - \alpha \left[\frac{(y^k \cdot y^k) - (n^k \cdot n^k)}{2} \right] > 0$$

and $\alpha > 0$. By Lemma 1.12, it can be shown that there is a solution to these inequalities iff for all $\lambda \in \mathbf{R}^k$ where $\lambda \geq 0$ and $\sum_{k=1}^K \lambda_k > 0$, we have $\sum_{k=1}^K \lambda_k (y^k - n^k) = 0$ implies $\sum_{k=1}^K \lambda_k \left[(y^k \cdot y^k) - (n^k \cdot n^k) \right] < 0$. Note that we may simply renormalize so that $\sum_k \lambda_k = 1$.

Remark 11.12 It can be similarly shown that a voting record V is strongly rationalizable by a *linear* preference iff for all $\lambda \in \mathbf{R}^k$ for which $\lambda \geq 0$ and $\sum_{k=1}^K \lambda_k = 1$, we have $\sum_{k=1}^K \lambda_k y^k \neq \sum_{k=1}^K \lambda_k n^k$. Thus, the condition of Euclidean rationalizability is strictly weaker than the condition of linear rationalizability. This should not be surprising, as a linear preference is like a Euclidean preference with an ideal point "at infinity." The situation would, of course, change if we allowed indifference into a voting record.

Section 11.2.1 also addressed the question of when the Euclidean model has no testable implications whatsoever. Instead of trying to describe the rationalizable datasets, we found properties on the parameters of the problems such that no data could refute the Euclidean model.

In the present context, we can ask a similar question. The difference compared to the environment in 11.2.1 is that now the policy positions are observed as vectors in $\mathbf{R}^d$. Let us define an *election* to be a binary set $\mathcal{E} = \{x,z\} \subseteq \mathbf{R}^d$, where $x \neq z$. For any finite collection of elections $\mathcal{E}_1, ..., \mathcal{E}_k$, a voting record V is *consistent* with this collection if for all $(x,z) \in V$, $\{x,z\} \in \mathcal{E}_i$ for some i, and for all i, if $\{x,z\} = \mathcal{E}_i$, then either $(x,z) \in V$ or $(z,x) \in V$. That is, a voting record is consistent with a sequence of elections iff it describes a sequence of possible votes over those elections. The next result is due to Degan and Merlo.

Theorem 11.13 *Suppose $k \leq d$. Then there is an open and dense set of election sequences of length k such that any consistent voting record is rationalizable by Euclidean preferences. If $d < k$, then for all election sequences there are voting records that are not rationalizable by Euclidean preferences.*

Proof. Let $\mathcal{E}$ be an election $\{x,z\}$. Let $\lambda \in \mathbf{R}^d$ be defined by $\lambda = x - z$ and let $c \in \mathbf{R}$ be defined by $c = \frac{(x \cdot x - z \cdot z)}{2}$. A simple calculation reveals that a Euclidean preference with ideal point y ranks $x \succ z$ if and only if $\lambda \cdot y > c$ and ranks $z \succ x$ iff $\lambda \cdot y < c$.

Each election $\mathcal{E}_i = \{x_i, z_i\}$ is identified with a pair (λ_i, c_i), defined as above. Given a voting record V, there is y such that the Euclidean preferences with ideal point y rationalize V iff there is $y \in \mathbf{R}^d$ such that $\lambda_i \cdot y > c_i$ if $(x_i, z_i) \in V$, and $\lambda_i \cdot y < c_i$ if $(z_i, x_i) \in V$. We may write in matrix form $\Lambda \in \mathbf{R}^{k \times d}$, where Λ collects the vectors λ_i, and $c \in \mathbf{R}^k$.

Now, with $k \leq d$, for an open and dense set of Λ, the range of the function $y \mapsto \Lambda \cdot y - c$ is k-dimensional. And by the preceding discussion, for any such Λ and c, and any V, there is y rationalizing V.

Suppose instead that $d < k$. We know then that $\lambda_1, ..., \lambda_k$ cannot be linearly independent; without loss of generality, let us suppose that $\lambda_k = \sum_{i=1}^{k-1} \alpha_i \lambda_i$, and let us suppose again without loss that $\alpha_i \geq 0$ for all i (we can always relabel x_i as z_i and z_i as x_i). Now consider the value $\overline{c} = \sum_{i=1}^{k-1} \alpha_i c_i$. There are two cases. Either $\overline{c} \leq c_k$ or $\overline{c} > c_k$. In the first case, whenever $\lambda_i \cdot y \leq c_i$ for all $i \leq k-1$, it follows that $\lambda_k \cdot y \leq c_k$. In the second case, whenever $\lambda_i \cdot y \geq c_i$ for all $i \leq k-1$, it follows that $\lambda_k \cdot y > c_k$. Either way, there are voting records that cannot be rationalized.

The message of Theorem 11.13 is similar to that of Theorem 11.6; it characterizes the environments where the Euclidean model is empirically vacuous.

11.3 CHAPTER REFERENCES

Theorem 11.1 is due to McGarvey (1953). Theorems 11.2 and 11.3 are due to Dushnik and Miller (1941). Theorem 11.4 is due to Ghouila-Houri (1962) and Gilmore and Hoffman (1964). The NP-completeness result is due to Yannakakis (1982). Results generalizing McGarvey's Theorem for other choice rules include Deb (1976) and Kalai (2004). Sprumont (2001) describes necessary and sufficient conditions for 2-unanimity rationalization by weak orders in certain economic environments. Theorem 11.5 appears in Fishburn (1969) without proof; a much more general statement can be found in Fishburn (1973a). Related to these results is the paper of Knoblauch (2005), which implicitly uses model-theoretic ideas (see Chapter 13) to describe the length of potential axiomatizations of Pareto rationalizability.

The results and proofs in Section 11.2.1 are due to Bogomolnaia and Laslier (2007). Bogomolnaia and Laslier (2007) actually establish a slightly different result than Theorem 11.6, that even when one admits linear preferences (preferences represented by a linear function), the result does not change.

Note that the function ρ described in the proof of Theorem 11.6 is allowed to depend on the profile $\succeq_1, ..., \succeq_n$. In general, we may not wish this to be the case. Bogomolnaia and Laslier (2007) discuss this issue.

As for a proof of Theorem 11.7, the construction of preferences $\succeq^*$ we suggest can be established as in Richter and Wong (2004).

Some related papers are useful to mention. Knoblauch (2010) provides a polynomial-time algorithm (and construction) for understanding when a profile is Euclidean rationalizable for $d = 1$. Azrieli (2011) studies Euclidean preferences with a "valence" dimension. Ballester and Haeringer (2011) describes conditions ensuring that a preference profile is rationalizable as a single-peaked profile for some ordering of the alternatives.

The discussion in Section 11.2.2 draws on Kalandrakis (2010) and Degan and Merlo (2009). Most of the technical ideas in this section can be ascribed to Richter and Wong (2004); we have followed Richter and Wong in the proof of Theorem 11.9, which appears in Kalandrakis (2010) with a different proof. Theorem 11.13 is due to Degan and Merlo (2009).

Richter and Wong (2004) deal with the following problem. Suppose we have a finite subset $K \subseteq \mathbf{R}^d$, and a complete and transitive binary relation $\succeq$ on K. When is it the case that there exists a complete, transitive, extension of $\succeq$ which can be represented by a concave utility function? (One difference with Kalandrakis (2010) is that he discards the completeness condition.)

Finally, Gomberg (2011, 2014) presents the testable implications of group behavior when the group can vary, and is assumed to be composed of rational individuals choosing according to a scoring rule.