CHAPTER 12

Revealed Preference and Systems of Polynomial Inequalities

It should be apparent by now that systems of linear inequalities emerge naturally in revealed preference theory. They constitute the essence of Afriat's Theorem, for example; and we formulated revealed preference problems using systems of linear inequalities in Chapters 3, 6, 7, and 8. In this chapter, we describe how revealed preference problems can generally be understood as a system of inequalities. From a purely computational perspective, one can very often solve a revealed preference problem by algorithmically solving the corresponding system of inequalities.

When the system of inequalities is linear, the problem is easy to solve both computationally and analytically. Here we develop a GARP-like acyclicity test (similar to the ones in Chapters 2 and 3). The test will follow from the linearity of the system of inequalities embodied in the revealed preference question.

We shall discuss an extension of the linear theory to systems of polynomial inequalities. The theory of polynomial inequalities will be seen to be very relevant for revealed preference theory (but harder to work with compared to the theory of linear inequalities).

12.1 LINEAR INEQUALITIES: THE THEOREM OF THE ALTERNATIVE AND REVEALED PREFERENCE

We start by revisiting the Theorem of the Alternative, or Farkas' Lemma from Chapter 1. It is easy to see why it is useful in revealed preference theory. We then discuss some sources of linear systems for popular models in economics. The following is a bit weaker than Lemma 1.13. It is written so as to emphasize that the lemma can be used to "remove existential quantifiers;" it states that an existential statement (one that start with "there is…") is equivalent to a universal statement (one that starts with "for all…"). Note that the discussion of the Tarski–Seidenberg Theorem in Chapter 9 is also about removing existential quantifiers.

Lemma 1.13′ *(Integer–Real Farkas) Let $\{A_i\}_{i=1}^{M}$ be a finite collection of vectors in $\mathbf{Q}^K$. The following statements are equivalent:*

I) *There exists* $y \in \mathbf{R}^K$ *such that for all* $i = 1, \ldots, M$, $A_i \cdot y > 0$.
II) *For all* $z \in \mathbf{Z}^M_+ \setminus \{0\}$, *it holds that* $\sum_{i=1}^M z_i A_i \neq 0$.

The vector y represents some unknown quantities. The vectors A_i encode some properties we want y to have. Typically, y is an unobservable theoretical object, such as a utility function. The properties in A_i can be theoretical (for example requiring utility to be monotonic) or come from the data, for example requiring the utility function to rationalize the data. In the example that we expand on below, M is the number of observations, and K is the number of possible alternatives from which an agent chooses. Then every observation corresponds to a vector A_i. For example, suppose that we observe object h chosen over object l. Then there is an A_i of the form $\mathbf{1}_h - \mathbf{1}_l$. See Section 12.1.2 below for more details. The existence of $y \in \mathbf{R}^K$ satisfying the inequalities then translates into the existence of a utility function rationalizing the data.

System I corresponds to the revealed preference formulation of a problem: A dataset is "rationalizable" if *there exists* a value for the theoretical object y that satisfies the right properties and explains the data. We call this an *existential* formulation of a theory. As stated, it is not falsifiable: If one is given a candidate solution y, it is easy to check whether System I is satisfied. If the candidate is a solution then we are done, but if the candidate y fails to satisfy System I then we have essentially no clue as to whether System I is solvable because there are infinitely many other potential solutions to the system. So when System I has no solution, and the dataset is "not rationalizable," then there is no way that checking individual vectors y one by one can tell us that the system is not solvable.

In contrast II is a *universal* statement. It says that something has to be true *for all* vectors z. Given a single $z \in \mathbf{Z}^M_+ \setminus \{0\}$ with $\sum_{i=1}^M z_i A_i = 0$, we know by Lemma 1.13′ that there cannot exist y solving System I. Despite the existential formulation of the theory, the theory is falsifiable. When System I has no solution, and the dataset is "not rationalizable," then no single y can certify that the theory has been falsified, but thanks to Lemma 1.13′, a single z can do that.

Lemma 1.13′ says a bit more about z. It says that it is a vector of integers. This is very often useful in finding a "combinatorial" revealed preference axiom, such as GARP or SARP.

Note that *verification* is, in some sense, a dual property to falsification. The existential formulation System I means that an individual y cannot certify that the system has no solution. But when the system does have a solution, a single vector y is enough to certify that a solution exists.

12.1.1 Linear systems from first-order conditions

First-order conditions are a common source of systems of inequalities in revealed preference theory. For example, in the revealed preference problem of rational demand theory, we obtain the system described in Afriat's Theorem from the first-order conditions in the consumer's maximization problem. The following is a heuristic derivation of the system of linear inequalities in Afriat's Theorem.

Recall the setup in Chapter 3. We observe a dataset (x^k, p^k), $k = 1, \ldots, K$. We want to know when there is a function $u : \mathbf{R}^n_+ \to \mathbf{R}$ such that x^k is a maximizer of u in the budget set $B(p^k, m^k) = \{x \in \mathbf{R}^n_+ : p^k \cdot x \leq m^k\}$, where $m^k = p^k \cdot x^k$.

If such a u exists, and if it is smooth and concave, then the first-order condition must be satisfied at the choices x^k:

$$\nabla u(x^k) - \lambda^k p^k = 0, \tag{12.1}$$

where $\nabla u(x^k)$ denotes the gradient of u at x^k, and λ^k is the Lagrange multiplier associated to this maximization problem. The unknowns here are the numbers $\nabla u(x^k)$ and λ^k.

We need to find values for these unknowns that are compatible with a concave u. Essentially we can solve for another set of numbers, *utility values*, $U^k = u(x^k)$, which can be extended to a function defined on all of $\mathbf{R}^n_+$. Concavity demands the following inequality be satisfied for all k and l:

$$U^l - U^k \leq \nabla u(x^k) \cdot (x^l - x^k). \tag{12.2}$$

Putting the equations of type (12.1), which result from the first-order conditions, together with inequalities of type (12.2) yields the linear system we solve for in Afriat's Theorem. By using Lemma 1.13 with this system one obtains GARP: see 3.1.1.

Another example can be obtained from Cournot oligopoly theory. Suppose that there are n firms in a market for a homogeneous good. The theory says that market price is determined as if each firm i has a cost function $c_i : \mathbf{R}_+ \to \mathbf{R}$, with $c_i(0) = 0$, and chooses quantity q_i to maximize

$$q_i P\left(\sum_{j=1}^{n} q_j\right) - c_i(q_i),$$

where $P : \mathbf{R}_+ \to \mathbf{R}_+$ is the market *inverse demand function*. The quantity $q_i P\left(\sum_{j=1}^n q_j\right)$ is the revenue of firm i, so the expression reflects that i maximizes profits.

Suppose that we observe K instances in which these firms were engaged in quantity competition. The data we observe are of the form:

$$(q_1^k, \ldots, q_n^k, p^k) : k = 1, \ldots K;$$

where $q_i^k \geq 0$ is the quantity chosen by firm i in instance k, and p^k is the prevailing market price. The functions c_i and P are *not* observable.

Now, say that a dataset is *Cournot rationalizable* if there are convex functions c_i and a smooth and decreasing function P such that q_i^k maximizes the profits of firm i, given that all other firms choose the quantities in the vector (q_j^k).

Now, the first-order condition for firm i is:

$$q_i^k P'\left(\sum_{j=1}^n q_j^k\right) + P\left(\sum_{j=1}^n q_j^k\right) - c_i'(q_i^k) = 0.$$

Recall that price p^k is observed as part of the data, so the unknowns are the numbers $P'(\sum_{j=1}^n q_j^k)$ and $c_i'(q_i^k)$. One needs to find cost and demand functions such that $p^k = P(\sum_{j=1}^n q_j^k)$, and derivatives behave properly. As with Afriat's Theorem, it is enough to do so by finding certain real numbers: in our case the numbers $P'(\sum_{j=1}^n q_j^k)$ and $c_i'(q_i^k)$, such that certain inequalities are satisfied.

We use the notation δ_i^k for the number $c'(q_i^k)$. The first-order condition implies that

$$\frac{\delta_j^k - p^k}{q_j^k} = \frac{\delta_i^k - p^k}{q_i^k} < 0, \tag{12.3}$$

for all i, j, and k, as $P'(\sum_j q_j^k)$ does not depend on i and must be negative for demand to slope down.

It is possible to show that data are Cournot rationalizable iff there are numbers δ_i^k, $i = 1,\ldots,N$, $k = 1,\ldots,K$ such that the linear inequalities (12.3) are satisfied, and such that when $q_i^k < q_i^l$ then $\delta_i^k < \delta_j^k$; the latter requirement captures that marginal cost must be monotone increasing for cost functions to be convex.

12.1.2 The existence of a rationalizing utility

We shall illustrate the role of linear inequalities with a very basic example, dealing with the simplest version of revealed preference for individual decision making.

Suppose that we are given a finite set X of alternatives. Suppose we are given a dataset in the form of an observed revealed preference relation $\succeq^R$. That is, we observe a set of binary comparisons, where an agent has chosen x over y iff $x \succeq^R y$. We want to know when R is rationalizable using a utility function $u : X \to \mathbf{R}$. Suppose that the relation R encompasses only strict comparisons, so $x \succeq^R y$ and $x \neq y$ implies that $x \succ^R y$. Then a utility function u rationalizes $\succeq^R$ if $u(x) > u(y)$ whenever $x \succ^R y$.

The problem can be set up as follows. A utility function is a vector in Euclidean space $\mathbf{R}^X$. Define a matrix B with $|X|$ columns. For every pair $(x,y) \in \succ^R$, include a row in B which has zeroes in all entries except for a 1 in the column for x and a -1 in the column for y. So the row is the vector $\mathbf{1}_x - \mathbf{1}_y$. Then there is a utility function that rationalizes the data if and only if there is a vector $u \in \mathbf{R}^X$ that solves the system $B \cdot u \gg 0$ of linear inequalities.[1]

[1] The matrix B is like the upper left submatrix of the matrix constructed in the proof of Afriat's Theorem.

By Lemma 1.13′ there is no rationalizing u (that is, no u such that $B \cdot u \gg 0$) if and only if there is $z > 0$ such that $z \cdot B = 0$. So suppose that that is the case, and let z be the vector promised by Lemma 1.13′. Let $(x,y) \in X \times X$ correspond to a row r for which $z_r > 0$; so $x \succ^R y$. There is at least one such pair because $z > 0$. There is a 1 in the column for x in row r. Now, no entry of z is negative, and $z \cdot B = 0$, so there must be some row r' with $z_{r'} > 0$ in which we have a -1 in the column for x. So there must exist x' with $x' \succ^R x$; the row r' corresponds to (x',x). Note that now x' is in the position that x was in when we started to make this argument. So there must exist $x'' \in X$ with

$$x'' \succ^R x' \succ^R x \succ^R y.$$

Since X is a finite set, by repeating the argument we must reach a cycle of $\succeq^R$. The argument shows that the order pair $\langle \succeq^R, \succ^R \rangle$ is not acyclic.

In conclusion, the non-existence of a solution to the system $u \cdot B \gg 0$ is equivalent to the existence of a cycle of $\succeq^R$. The standard result on acyclicity therefore emerges as a consequence of Lemma 1.13.

The same reasoning can be applied to many of the results in Chapter 2. We sketch proofs of Theorems 2.6, 2.8, 2.16, and 2.17 using this device. In fact, the Theorem of the Alternative is often a useful way of "guessing" the appropriate concept of rationalization specified by a particular choice-theoretic axiom.

Consider Theorem 2.6. Suppose that X is a finite set. Rationalizability in the sense of Theorem 2.6 is equivalent to the existence of a function $u : X \to \mathbf{R}$ for which $x \succeq^c y$ implies $u(x) \geq u(y)$ and $x \succ^c y$ implies $u(x) > u(y)$. One can then proceed almost exactly as we did in the previous paragraphs to obtain the acyclicity result. The only difference is that we now have weak revealed preference comparisons. Let the order pair $\langle \succeq^c, \succ^c \rangle$ be as defined in Chapter 2. Construct a matrix B with $|X|$ columns, and a row for each $\succeq^c$ or $\succ^c$ comparison. For comparisons of the type $x \succeq^c y$, we add a row to the matrix of form $\mathbf{1}_x - \mathbf{1}_y$; likewise for comparisons of type $x \succ^c y$. We want to find a solution $u \in \mathbf{R}^X$ such that for rows B_i of type $\succeq^c$, we have $B_i \cdot u \geq 0$, and for rows of type $\succ^c$, we have $B_i \cdot u > 0$. Applying Lemma 1.13 and using similar techniques as in the preceding proof for removing cycles, we come up with congruence as the necessary and sufficient condition.

Theorem 2.8 is a bit different. We can no longer work with a utility function, as it is clear that not every complete binary relation can be so encoded (unless it is transitive, of course). Instead, we encode the preference via a function $\varphi : X^2 \to \mathbf{R}$ for which $\varphi(x,y) = -\varphi(y,x)$. The interpretation here is that $x \succeq y$ iff $\varphi(x,y) \geq 0$. This suggests a matrix B, whose rows are indexed by X^2, and for each pair x,y, there is a row $\mathbf{1}_{(x,y)} + \mathbf{1}_{(y,x)}$. Now, we also add a row for each comparison of the type $x \succeq^c y$; namely, this row consists of $\mathbf{1}_{(x,y)}$; likewise, we add a row for each comparison of the type $x \succ^c y$, and again, this row consists of $\mathbf{1}_{(x,y)}$. We want to find a solution $\varphi \in \mathbf{R}^{X^2}$ such that for rows B_i of the first type, we have $B_i \cdot \varphi = 0$, for rows B_i of the second type, we have $B_i \cdot \varphi \geq 0$, and for rows B_i of the third type, we have $B_i \cdot \varphi > 0$. Again, a simple application of the Theorem of the Alternative leads to the weak axiom of revealed preference.

Theorems 2.16 and 2.17 can be proved using similar techniques to the preceding – the distinction is that in the case of Theorem 2.16, we search for a $u \in \mathbf{R}^X$ for which $x \succ^c y$ implies $u(x) > u(y)$ (no requirement is made for $x \succeq^c y$), and in the case of Theorem 2.17, we search for a $\varphi \in \mathbf{R}^{X^2}$ for which $x \succ^c y$ implies $\varphi(x,y) > 0$ (again no requirement is made for $x \succeq^c y$).

The general form of the requirement in these theorems is something along the following lines: "*There exists* some function such that *for all* pairs, the function is related to revealed preference in some way." The existential statement quantifies an unobservable object. The universal (for all) statement quantifies observable objects. The Theorem of the Alternative is used to show that these statements are logically equivalent to a statement which is universal and stated only in terms of observable objects; for example, Theorem 2.6 states that existence of such a utility is equivalent to the statement: "For all $x_1, \ldots, x_n$, it is not the case that for $i = 1, \ldots, n-1$, $x_i \succeq^c x_{i+1}$ and $x_n \succ^c x_1$." Though existential statements are in principle problematic from the viewpoint of falsifiability, we see that in general this might not be true when the existential operator quantifies unobservable objects. Existential operators operating on unobservables can sometimes be "removed," and turned into an equivalent universal statement involving only observables.

12.2 POLYNOMIAL INEQUALITIES: THE POSITIVSTELLENSATZ

Some revealed preference problems give rise to nonlinear, specifically polynomial, systems of inequalities. Below we give a detailed example taken from Nash bargaining theory (see Chapter 10). For polynomial systems of inequalities, there is a result like the Theorem of the Alternative; it is called the *Positivstellensatz*.

In order to state the Positivstellensatz, we need a bit of notation. Given is a collection of variables, say $\{x_1,...,x_n\}$. We assume that the notion of a polynomial is understood. We will describe one version of the Positivstellensatz.

Given a collection of polynomials $\{f_1,...,f_m\}$ over the variables $\{x_1,...,x_n\}$, we define the *ideal* of $\{f_1,...,f_m\}$ to be the collection of all polynomials which can be written in the form:

$$\sum_{i=1}^{m} g_i f_i,$$

where g_i is a polynomial. We define the *cone* generated by $f_1,...,f_m$ to be the smallest set of polynomials that (1) includes all sums of squares of polynomials, (2) includes all polynomials $f_1,...,f_m$, and (3) is closed under addition and multiplication. It is easy to see that any such element can be written as

$$\sum_{S \subseteq \{1,...,m\}} \left(g_S \prod_{i \in S} f_i \right),$$

where g_S is a sum of squares of polynomials. Finally, we define the *multiplicative monoid* generated by $f_1,...,f_m$ to be the collection of polynomials of the form $\prod_{i=1}^{m} f_i^{a_i}$, where each a_i is a non-negative integer.

Let f_i, g_j, and h_l be polynomials over $\{x_1,\dots,x_n\}$, for $i=1,...,m$, $j=1,...,k$, and $l=1,...,q$.

Theorem 12.1 (Positivstellensatz). *A collection of inequalities $f_i(x)=0$, $i=1,...,m$, $g_i(x)\geq 0$, $i=1,...,k$, $h_l(x)\neq 0$, $i=1,...,q$ has no solution iff there exist polynomials f in the ideal of $\{f_1,...,f_m\}$, g in the cone generated by $\{g_1,...,g_k\}$, and h in the multiplicative monoid generated by $\{h_1,...,h_q\}$ for which $f+g+h=0$.*

The Positivstellensatz thus provides an alternative system of polynomial inequalities, such that the first system has no solution iff the second system has a solution. The statement is similar in spirit to the Theorem of the Alternative, but the unknowns in the second system are polynomials, not real numbers. From a computational perspective, the problem is clearly harder to deal with. Conceptually, the Positivstellensatz has implications that are similar to those we highlighted for the Theorem of the Alternative in Section 12.1: it turns an existential, verifiable theory into a universal, falsifiable statement. Thus, if a system of polynomial inequalities cannot be satisfied, it is possible to demonstrate, or certify, its infeasibility.

To actually find a falsification is challenging computationally. One practical approach is to search for a solution to the second system of inequalities among polynomials of bounded degree. This approach is common in the engineering literature (see the references in Section 12.3). So one would look for f in the ideal of $\{f_1,...,f_m\}$, g in the cone generated by $\{g_1,...,g_k\}$, and h in the multiplicative monoid generated by $\{h_1,...,h_q\}$ for which $f+g+h=0$; but restricting such a search to polynomials of degree smaller than some given bound. The search for a solution among polynomials of bounded degree can be formulated as a semidefinite program, and thus there are efficient algorithms for solving the problem.

Of course, the approach only works when one finds a solution: when one finds polynomials certifying a solution $f+g+h=0$, thus certifying that the first system of inequalities (the one we are really interested in) has no solution. If one does not find a solution among polynomials of bounded degree, then it is possible that there is a solution of a higher degree, or that the original system in fact has a solution.

12.2.1 Application: Nash bargaining

To get a sense for how these ideas might be applied in economics, let us consider the problem of testing Nash bargaining theory, as in Section 10.3 of Chapter 10. In Chapter 10, we assumed that the disagreement point was fixed across observations, and that it was the same for all agents. If we relax that assumption then things get more complicated. We can still formulate the problem as a polynomial system of inequalities and use the Positivstellensatz to obtain an answer. The alternative, or dual, system from the Positivstellensatz gives a test for Nash bargaining theory.

Assume a finite set N of agents is given. Generalizing the setup from Section 10.3, suppose that a dataset is a set $D = \{(x^k, d^k) : k = 1, \ldots, K\}$; where for each k, x^k and d^k are vectors in $\mathbf{R}^N_+$ such that $x^k \gg d^k \geq 0$. These are observations of K bargaining instances; in the kth instance, agent i received x^k_i dollars, while his disagreement outcome was d^k_i.

A dataset is *Nash bargaining rationalizable* if there are concave and monotonic functions $u_i : \mathbf{R}_+ \to \mathbf{R}$, $i \in N$, such that

$$\prod_{i=1}^{n} \left[u_i(x^k_i) - u_i(d^k_i)\right] \geq \prod_{i=1}^{n} \left[u_i(x_i) - u_i(d^k_i)\right],$$

for all $x \in \mathbf{R}^N_+$ such that $x_i \geq d^k_i$ and $\sum_{i \in N} x_i \leq \sum_{i \in N} x^k_i$.

Proposition 12.2 *A dataset D is Nash rationalizable if and only if for all $i \in N$, there are numbers $U_i(d^k_i), U_i(x^k_i), M_i(d^k_i)$, and $M_i(x^k_i)$ for $k = 1, \ldots, K$ which solve the following equations: for all $z \in \{x^k_i\}$, $z' \in \{x^k_i, d^k_i\}$, and all i, j, and k,*

$$\frac{M_i(x^k_i)}{U_i(x^k_i) - U_i(d^k_i)} = \frac{M_j(x^k_j)}{U_j(x^k_j) - U_j(d^k_j)}$$

and for all $z, z' \in \bigcup_{i=1}^{K} \{d^k_i, x^k_i\}$

$$\begin{cases} U_i(z) - U_i(z') > 0 & \text{if } z < z', \\ M_i(z')(z - z') \geq U_i(z) - U_i(z'). & \end{cases}$$

Proposition 12.2 is straightforward. We simply ask for numbers $U_i(z)$ to signify levels of utility, and $M_i(z)$ for supergradients, or marginal utilities. The first system of equalities ensures that the first-order conditions for the maximization of the Nash product hold. The second set of inequalities make sure that utility is monotonic and that marginal utilities are a supergradient of the utility.

Proof. We first show that if we are given increasing and concave utility functions u_i, then $x^k_1, \ldots, x^k_n$ is a solution to $\max_{\sum_{i \in N} x_i = M} \prod_{i \in N} [u_i(x_i) - u_i(d^k_i)]$ if and only if for each i, there is a supergradient μ_i of u_i at x^k_i for which

$$\frac{\mu_i}{U_i(x^k_i) - U_i(d^k_i)} = \frac{\mu_j}{U_j(x^k_j) - U_j(d^k_j)}.$$

To this end, define $\mathcal{U} = \{\left(u_1(x_1) - u_1(d^k_1), \ldots, u_n(x_n) - u_n(d^k_n)\right) : \sum_{i \in N} x_i = M, x_i \geq d^k_i\}$, and consider maximizing the function $f(y) = \prod_{i \in N} y_i$ subject to $y \in \mathcal{U}$. A point $u \in \mathcal{U}$ maximizes f if and only if $\left(\prod_{i \neq 1} u_i, \ldots, \prod_{i \neq n} u_i\right)$ supports $\mathcal{U}$ at u (by definition). Because f is strictly convex, and since $\mathcal{U}$ is convex and compact, there is a unique such maximizer u^*. It is clear that $u_i^* > 0$ for all $i \in N$.

This states that there is a unique solution $x_1^k, \ldots, x_n^k$ to the Nash problem for which $u_i(x_i^k) - u_i(d_i^k) = u_i^*$. We define $\lambda_j = \prod_{i \neq j}[u_i(x_i^k) - u_i(d_i^k)]$. We know that $\sum_{i \in N} \lambda_i u_i(x_i)$ is maximized at $x_1^k, \ldots, x_n^k$ across all x_i for which $\sum_{i \in N} x_i = M$. Our next step is to show that this can happen if and only if the vector $(1/\lambda_1, \ldots, 1/\lambda_n)$ is proportional to a vector of supergradients.

Since the constraints $x_i \geq d_i^k$ are not binding, we can set up the Lagrangian for the problem, say $L(x, \mu) = \sum_{i \in N} \lambda_i u_i(x_i) + \mu(M - \sum_{i \in N} x_i)$, and note that it is equal to $L(x, \mu) = \sum_{i \in N}[\lambda_i u_i(x_i) - \mu x_i] + \mu M$. We know the constraint $\sum_{i \in N} x_i = M$ is binding, so that the solution to the maxmin problem features $\mu^* > 0$. For μ^*, we know that $\max_x L(x, \mu^*)$ is equal to the maximum Nash product subject to the constraint, and has the same solution. This is equivalent to saying that $(\lambda_i/\mu^*) u_i(x_i^k) - x_i^k \geq (\lambda_i/\mu^*) u_i(x_i) - x_i$ for all x_i, or, rewriting:

$$u_i(x_i) + (\mu^*/\lambda_i)(x_i^k - x_i) \leq u_i(x_i^k).$$

This is equivalent to saying that μ^*/λ_i is a supergradient, or that the vector $(1/\lambda_1, \ldots, 1/\lambda_n)$ is proportional to a supergradient.

Another way of saying that $(1/\lambda_1, \ldots, 1/\lambda_n)$ is proportional to a vector of supergradients is saying that for all $i \in N$, there is a supergradient $M_i(x_i^k)$ of u_i at x_i^k such that for all i,j, $\frac{\lambda_i}{\lambda_j} = \frac{M_j(x_j^k)}{M_i(x_i^k)}$. Writing out the explicit form of λ and eliminating terms, this is equivalent to saying that $\frac{M_i(x_i^k)}{u_i(x_i^k) - u_i(d_i^k)} = \frac{M_j(x_j^k)}{u_j(x_j^k) - u_j(d_j^k)}$, which is precisely the condition in the theorem. The other conditions simply say that M_i is a supergradient, and that u_i is strictly increasing.

Conversely, the details of how to construct a utility function from these numbers essentially follow from Afriat, defining $u_i(x) = \inf_{z \in \bigcup_{k=1}^K \{x_i^k, d_i^k\}} U_i(z) + M_i(z)(x - z)$, where the infimum is taken over all data points. It is then simple to verify by construction that for all $z \in \bigcup_{k=1}^K \{x_i^k, d_i^k\}$, $M_i(z)$ is a supergradient of u_i at z. From this, the fact that the equality in the statement of the theorem is solved implies that the Nash product is maximized for this collection of utility functions (by the previous argument).

From Proposition 12.2 and the Positivstellensatz we can obtain a test for Nash bargaining. The conditions in the proposition involve polynomials in $4|N|K$ variables, namely $U_i(d_i^k), U_i(x_i^k), M_i(d_i^k)$, and $M_i(x_i^k)$. We have the system of polynomial equalities

$$\frac{M_i(x_i^k)}{U_i(x_i^k) - U_i(d_i^k)} = \frac{M_j(x_j^k)}{U_j(x_j^k) - U_j(d_j^k)}$$

and the inequalities for all $z, z' \in \bigcup_{i=1}^K \{d_i^k, x_i^k\}$,

$$\begin{cases} U_i(z) - U_i(z') > 0 & \text{if } z < z', \\ M_i(z')(z - z') \geq U_i(z) - U_i(z'). \end{cases}$$

12.3 CHAPTER REFERENCES

In general, Lemma 1.13 allows us to find the exact empirical content of many linear models. Scott (1964) is a classic reference. The issue with Lemma 1.13 is that the universal quantifier does not typically operate on observables (here, z is simply a vector – but in our example, observed data were revealed preference comparisons). It turns out though, that since z is integer-valued, this universal quantifier can be translated directly into observables. For example, in the revealed preference example in Section 12.1.2, the fact that for all for all $z \in \mathbf{Z}^K_+$ with $\sum_{i=L+1}^K z_i > 0$, we have $\sum_{i=1}^K z_i A_i \neq 0$ is the same as saying there are no preference cycles. In fact, it is often the case that one can require the universal quantifier on z to operate over a *finite* number of z.

The discussion of the Cournot model in Section 12.1.1 borrows from the paper of Carvajal, Deb, Fenske, and Quah (2013). These authors prove that the condition we have stated (the existence of a solution to the system of inequalities) is equivalent to Cournot rationalizability.

The discussion of existential and universal axioms is inspired by the work of Popper (1959). Popper claimed famously that existential theories are not falsifiable: consider the theory that claims that there is a black swan. No matter how many non-black swans one observes, they do not disprove the theory. Universal theories are falsifiable, for example the theory that claims that all swans are white. The observation of a single non-white swan would disprove the theory. The papers by Van Benthem (1976) and Chambers, Echenique, and Shmaya (2012) present a general result on how existential quantifiers over theoretical objects can be removed, and a theory may be shown to be falsifiable. See also Craig and Vaught (1958), Lemma 3.4 for a related result in a more restrictive environment.

The Positivstellensatz as formulated in Theorem 12.1 can be found, for example, in Bochnak, Coste, and Roy (1998), Theorem 4.4.2. It is originally due to Krivine (1964) and rediscovered by Stengle (1974). It is interesting that, while many economists know and apply the Theorem of the Alternative, there are almost no applications of the Positivstellensatz (Richter, 1975 is a notable exception, which itself builds on the work of Tversky, 1967). The theorem is potentially useful to applied economists, who would use the algorithms in Parrilo (2004) to carry out tests on actual datasets.

The Positivstellensatz is closely related to, but distinct from, the Tarski–Seidenberg Theorem; see our discussion in Chapter 9. Brown and Matzkin (1996) (see also Brown and Kubler (2008)) exploit this technique to find testable implications of equilibrium behavior. They also explain how the well-known equivalence of the strong axiom of revealed preference and rationality is a special case of Tarski–Seidenberg.

The approach of searching among polynomials of bounded degree is described in Parrilo (2003). Indeed, it can be shown to revert to a classical semidefinite programming problem. This approach is outlined in Parrilo (2003), a shorter introduction is provided in Parrilo (2004); see especially

Example 1 there. Marshall (2008), Chapter 10 provides a detailed explanation. Also see Sturmfels (2002).

Proposition 12.2 is from Chambers and Echenique (2014b). There are similar results in Carvajal and González (2014) and Cherchye, Demuynck, and De Rock (2013). Cherchye, Demuynck, and De Rock (2013) propose a different computational approach than the one we have emphasized here. They propose instead using integer programming to obtain a test for Nash bargaining.