CHAPTER 1

Mathematical Preliminaries

We begin by reviewing basic mathematical facts and notation that we shall use repeatedly in the book. The basic results we need relate to binary relations and their extensions and representations, and to solutions to systems of inequalities.

1.1 BASIC DEFINITIONS AND NOTATIONAL CONVENTIONS

1.1.1 Relations

Let X be a set. An *n-ary relation on* X is a subset of X^n. A *binary relation on* X is a subset of $X \times X$. When R is a binary relation on X, we write $(x,y) \in R$ as $x\,R\,y$. When R is an n-ary relation, we also write $R(x_1,\ldots,x_n)$ instead of $(x_1,\ldots,x_n) \in R$.

Given a binary relation R, define its *strict part*, or *asymmetric part*, P_R by $(x,y) \in P_R$ iff $(x,y) \in R$ and $(y,x) \notin R$. Define its *symmetric part*, or *indifference relation*, I_R by $(x,y) \in I_R$ iff $(x,y) \in R$ and $(y,x) \in R$.

Two elements $x,y \in X$ are *unordered* by R if $(x,y) \notin R$ and $(y,x) \notin R$. Two elements are *ordered* by R when they are not unordered by R. We say that a binary relation B is an *extension* of R if $R \subseteq B$ and $P_R \subseteq P_B$. A binary relation B is a *strict extension* of R if it is an extension, and in addition there is a pair that is unordered by R but ordered by B. Finally, a binary relation is *complete* if it leaves no pair of elements unordered. That is, R is complete if for all x and y, $x\,R\,y$ or $y\,R\,x$ (or both).

The following are standard properties of binary relations. A binary relation R is:

- *transitive* if, for all x, y, and z, $x\,R\,y$ and $y\,R\,z$ imply that $x\,R\,z$;
- *quasitransitive* if, for all x, y, and z, $x\,P_R\,y$ and $y\,P_R\,z$ imply that $x\,P_R\,z$;
- *reflexive* if $x\,R\,x$ for all x;
- *irreflexive* if $(x,x) \notin R$ for all x;
- *symmetric* if, for all x and y, $(x,y) \in R$ implies that $(y,x) \in R$;
- *antisymmetric* if, for all x and y with $x \neq y$, $(x,y) \in R$ implies that $(y,x) \notin R$.

Observe that if a binary relation is complete, then it is reflexive.

The following are types of binary relations: A binary relation is a(n)

- *weak order* if it is complete and transitive;
- *linear order* if it is complete, transitive, and antisymmetric;
- *equivalence relation* if it is reflexive, symmetric, and transitive;
- *partial order* if it is reflexive, transitive, and antisymmetric.

1.1.2 Partially ordered sets

If $\geq$ is a partial order on X, an *upper bound* on $A \subseteq X$ (with respect to $\geq$) is an element $x \in X$ such that $x \geq y$ for all $y \in A$; a *lower bound* on $A \subseteq X$ is an element $x \in X$ such that $y \geq x$ for all $y \in A$. A *least upper bound* on A is an upper bound x on A such that if y is an upper bound on A then $y \geq x$. A *greatest lower bound* on A is a lower bound x on A such that if y is a lower bound on A then $x \geq y$. We write $x \vee y$, called the *join* of x and y, for the least upper bound on the set $\{x, y\}$; and $x \wedge y$, called the *meet* of x and y, for the greatest lower bound on the set $\{x, y\}$. A set X, with a partial order $\geq$ on X, is a *lattice* if, for all x and y in X, $x \vee y$ and $x \wedge y$ are defined in X with respect to $\geq$.

Let X, with the partial order $\geq$, be a lattice. A function $u : X \to \mathbf{R}$ is *supermodular* if for all $x, y \in X$,

$$u(x) + u(y) \leq u(x \vee y) + u(x \wedge y),$$

and *submodular* if $-u$ is supermodular.

If $\geq$ is a partial order on X, a *chain* is a subset $Y \subseteq X$ such that for all $x, y \in Y$, either $x \geq y$ or $y \geq x$. The following result, known as *Zorn's Lemma*, is one of the many equivalents of the axiom of choice in set theory. It allows one to extend many induction-style proofs to sets of arbitrary cardinality. We shall use it several times.

Zorn's Lemma *Let $\geq$ be a partial order on a set X, and suppose that for every chain $Y \subseteq X$ there is an upper bound of Y in X (with respect to $\geq$). Then there exists $x^* \in X$ such that, for all $x \in X$, $x \geq x^*$ implies $x = x^*$.*

An element like x^* in the statement of Zorn's Lemma, with the property that $x \geq x^*$ implies $x = x^*$, is a *maximal element* for $\geq$ in X.

1.1.3 Euclidean spaces

We use $\mathbf{R}$ to denote the real numbers, $\mathbf{Q}$ to denote the rational numbers, and $\mathbf{Z}$ to denote the integers. When $X = \mathbf{R}$, we have the following familiar binary relation: $> \subseteq \mathbf{R}^2$ defined as $x > y$ if there is some $z \neq 0$ such that $x = y + z^2$. The relation $\geq$ is defined by $x \geq y$ if either $x = y$ or $x > y$. We often write $x > y$ as $y < x$, and $x \geq y$ as $y \leq x$. For any $x, y \in \mathbf{R}$ with $x < y$, the open interval $\{z \in \mathbf{R} : x < z < y\}$ is denoted by (x, y).

Let $X = \mathbf{R}^n$; $x_i \in \mathbf{R}$ is the ith *component* of the *vector* x, for $i = 1, \ldots, n$. The *inner product* of two vectors x and y is $x \cdot y = \sum_{i=1}^n x_i y_i$. Define the binary relation $\geq$ by $x \geq y$ if $x_i \geq y_i$ for $i = 1, \ldots, n$. Define $>$ and $\gg$ by $x > y$ if $x \geq y$ and $x \neq y$; and $x \gg y$ if $x_i > y_i$ for $i = 1, \ldots, n$. Denote by $\mathbf{R}^n_+$ the set of $x \in \mathbf{R}^n$ with $x \geq 0$; and by $\mathbf{R}^n_{++}$ the set of $x \in \mathbf{R}^n$ with $x \gg 0$.

Note that $\geq$ on $\mathbf{R}^n$ is a partial order, while $\geq$ on $\mathbf{R}$ is a linear order. The set $\mathbf{R}^n$ with the order $\geq$ is a lattice, and for any $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ we have $x \vee y = (\max\{x_i, y_i\})_{i=1}^n$ and $x \wedge y = (\min\{x_i, y_i\})_{i=1}^n$.

The usual (Euclidean) norm on $\mathbf{R}^n$ is $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$. The *interior* of a set $A \subseteq \mathbf{R}^n$ is the set of $x \in A$ for which there is $\varepsilon > 0$ such that if $\|x - y\| < \varepsilon$ then $y \in A$. The interior of A is denoted by $\operatorname{int} A$. We say that x is interior to A if $x \in \operatorname{int} A$.

We use standard notational conventions for functions and correspondences. Let X and Y be sets. We denote by $u : X \to Y$ a function with domain X and range a subset of Y. A *correspondence* is a function $\varphi : X \to 2^Y$, which we denote by $\varphi : X \rightrightarrows Y$. When $X \subseteq \mathbf{R}^n$, $Y = \mathbf{R}$, $x \in X$, and $u : X \to Y$ is differentiable (the definition of which is omitted), then $\nabla u(x)$ denotes the *gradient* of u at x.

We use $\mathbf{1}$ to denote an *indicator function*: for $E \subseteq X$, $\mathbf{1}_E : X \to \mathbf{R}$ is the function $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \notin E$; $\mathbf{1}_x$ denotes $\mathbf{1}_{\{x\}}$.

The notation for indicator functions is also used as an *indicator vector*. When $E \subseteq \{1, \ldots, n\}$, $\mathbf{1}_E$ is the vector in $\mathbf{R}^n$ that has a 1 in the ith component if $i \in E$, and has a 0 in the ith component if $i \notin E$. The vector $\mathbf{1}_i$ is called the ith *unit vector*. Finally, we write $\mathbf{1}$ for $\mathbf{1}_{\{1,\ldots,n\}}$, the vector all of whose components are 1.

1.2 PREFERENCE AND UTILITY

A *(rational) preference relation* on X is a binary relation $\succeq \subseteq X \times X$ that is a weak order. A preference relation that is a linear order is called a *strict preference* or a *strict preference relation*.

When we use the $\succeq$ notation for a binary relation, we write $\succ$ for $P_{\succeq}$, its strict part, and $\sim$ for its symmetric part $I_{\succeq}$. Note that when $\succeq$ is a preference relation then $\succ$ is asymmetric and transitive, while $\sim$ is an equivalence relation.

1.2.1 Properties of preferences

For a preference relation $\succeq$ on $X \subseteq \mathbf{R}^n$, we can define the following standard properties. We say that $\succeq$ is:

- *locally nonsatiated* if for all x and $\varepsilon > 0$, there is some $y \in X$ such that $\|x - y\| < \varepsilon$ and $y \succ x$;
- *monotonic* if $x \gg y$ implies $x \succ y$;
- *strictly monotonic* if $x > y$ implies $x \succ y$;
- *continuous* if the sets $\{z \in X : z \succeq x\}$ and $\{z \in X : x \succeq z\}$ are relatively closed in X for every x.

The set $\{z \in X : z \succeq x\}$ is the *upper contour set* of $\succeq$ at x: the set of elements that are at least as good as x. The *strict upper contour set* of $\succeq$ at x is $\{z \in X : z \succ x\}$. And the set $\{z \in X : x \succeq z\}$ is the *lower contour set* of $\succeq$ at x.

In the case when $X \subseteq \mathbf{R}^n$ is a convex set, it is often useful to understand the relation between a preference and convex combinations of elements in X. Say that a binary relation $\succeq$ is

- *convex* if $\lambda \in (0,1)$, $y \succeq x$, and $z \succeq x$ implies that $\lambda y + (1-\lambda)z \succeq x$;
- *strictly convex* if $\lambda \in (0,1)$, $y \succeq x$, $z \succeq x$, and $y \neq z$ implies that $\lambda y + (1-\lambda)z \succ x$.

So a preference relation $\succeq$ is convex if the upper contour set of $\succeq$ at x is a convex set, for all $x \in X$.

1.2.2 Utility

Preference relations are often described through a *utility function* $u : X \to \mathbf{R}$ by $x \succeq y$ iff $u(x) \geq u(y)$. We say that u *represents* $\succeq$. In that case we say that u is locally nonsatiated, monotonic, or strictly monotonic if the preference relation it represents has those properties. A utility function is *quasiconcave* if the preference relation it represents is convex; and *strictly quasiconcave* if the preference relation it represents is strictly convex. Additionally, we say that u is continuous or concave if it satisfies those properties according to its ordinary definitions. A *smooth* utility is one for which all partial derivatives of all orders exist.

We have the following general theorem.

Theorem 1.1 *A preference relation has a utility function that represents it iff there exists an (at most) countable set $Z \subseteq X$ with the following property: for all $x, y \in X$ for which $x \succ y$, there exists $z \in Z$ for which $x \succeq z \succeq y$.*

Proof. First, we show that if there exists such a utility representation, then there exists a countable set $Z \subseteq X$ satisfying the property in the statement of the theorem.

We shall exhibit a two-part construction. Let $A = u(X) \subseteq \mathbf{R}$, a nonempty set of real numbers. Consider two countable sets of open intervals. The first, denoted by $\mathcal{I}$, is the set of all open intervals with rational endpoints. The second, denoted by $\mathcal{I}'$, is the set of open intervals (x,y) with $x, y \in A$ and $(x,y) \cap A = \varnothing$. We show that $\mathcal{I}'$ is countable by the following argument: For $(x,y) \in \mathcal{I}'$, choose $q_{(x,y)} \in \mathbf{Q}$ such that $x < q_{(x,y)} < y$.[1] Note that, since any two intervals in $\mathcal{I}'$ are disjoint, the mapping $(x,y) \mapsto q_{(x,y)}$ is one-to-one. Since $\mathbf{Q}$ is countable, $\mathcal{I}'$ is countable.

Now, for each $I \in \mathcal{I}$, if $I \cap A \neq \varnothing$, pick $a \in I \cap A$ and denote this by a_I. Denote the set $\mathcal{A} = \{a_I : I \in \mathcal{I}, I \cap A \neq \varnothing\}$, and note that this set is at most countable

[1] This does not require the axiom of choice. Enumerate $\mathbf{Q} = \{q_1, q_2, \ldots\}$, and choose $q_{(x,y)}$ to be the element of $\mathbf{Q}$ for which $x < q_{(x,y)} < y$ associated with the lowest index in the enumeration.

because $\mathcal{I}$ is countable. Define $\mathcal{A}'$ to be the set of all x for which $(x,y) \in \mathcal{I}'$, for some y. Note that $\mathcal{A}'$ is at most countable because $\mathcal{I}'$ is.

Using the axiom of choice, for each $a \in \mathcal{A} \cup \mathcal{A}'$, choose one element $x(a) \in X$ such that $u(x(a)) = a$. Let $Z = \{x(a) : a \in \mathcal{A} \cup \mathcal{A}'\}$. It is clear that Z is at most countable and satisfies the requisite property.

Conversely, suppose that there exists an at most countable set Z satisfying the property in the hypothesis. Suppose that Z is countable, and label it as $\{z_k\}_{k=1}^{\infty}$. Define the number $u_k = 1/2^k$. Clearly, $\sum_{k=1}^{\infty} u_k < \infty$. For all $x \in X$, define $u(x) = \sum_{\{k:x \succ z_k\}} u_k - \sum_{\{k:z_k \succ x\}} u_k$. To verify that it is indeed a utility representation, first suppose that $x \succeq y$. Then $\{k : y \succ z_k\} \subseteq \{k : x \succ z_k\}$, so that $\sum_{\{k:y \succ z_k\}} u_k \leq \sum_{\{k:x \succ z_k\}} u_k$, and $\{k : z_k \succ x\} \subseteq \{k : z_k \succ y\}$, so that $\sum_{\{k:z_k \succ x\}} u_k \leq \sum_{\{k:z_k \succ y\}} u_k$. We conclude that $\sum_{\{k:y \succ z_k\}} u_k - \sum_{\{k:z_k \succ y\}} u_k \leq \sum_{\{k:x \succ z_k\}} u_k - \sum_{\{k:z_k \succ x\}} u_k$, or $u(y) \leq u(x)$.

For the strict preference, suppose that $x \succ y$. By the property of Z, there exists some z_k such that $x \succeq z_k \succeq y$. Then by completeness and transitivity, either $x \succ z_k$ or $z_k \succ y$. If the former, then $\{j : y \succ z_j\} \subsetneq \{j : x \succ z_j\}$, so that $\sum_{\{j:y \succ z_j\}} u_j < \sum_{\{j:x \succ z_j\}} u_j$. If the latter, then $\{j : z_j \succ x\} \subsetneq \{j : x \succ z_j\}$, so that $\sum_{\{j:z_j \succ x\}} u_j < \sum_{\{j:z_j \succ y\}} u_j$. In either case, we conclude that $\sum_{\{j:y \succ z_j\}} u_j - \sum_{\{j:z_j \succ y\}} u_j < \sum_{\{j:x \succ z_j\}} u_j - \sum_{\{j:z_j \succ x\}} u_j$, or $u(x) > u(y)$.

The proof is similar in the case where Z is finite.

1.3 ORDER PAIRS, ACYCLICITY, AND EXTENSION THEOREMS

An *order pair* on X is a pair of binary relations $\langle R, P \rangle$ such that $P \subseteq R$. If R is a binary relation, then $\langle R, P_R \rangle$ is an order pair, but we shall encounter order pairs $\langle R, P \rangle$ in which P may not be the strict part of R. An order pair $\langle R', P' \rangle$ is an *order pair extension* of $\langle R, P \rangle$ if $R \subseteq R'$ and $P \subseteq P'$. (Our previous definition of when a binary relation B is an extension of R corresponds to $\langle R, P_R \rangle$ being an order pair extension of $\langle B, P_B \rangle$.)

An order pair $\langle R, P \rangle$ is *acyclic* if there is no sequence $x_1, x_2, \ldots, x_L$ such that

$$x_1 \, R \, x_2 \, R \ldots R \, x_L,$$

and $x_L \, P \, x_1$. A sequence in the situation above is called a *cycle* of $\langle R, P \rangle$, and L is the *length of the cycle*. A single binary relation R is acyclic if the order pair $\langle R, R \rangle$ is acyclic.

Observation 1.2 *If $\langle R, P \rangle$ is acyclic, then $\langle R \cup (x,x), P \rangle$ is acyclic for any $x \in X$. Hence, any acyclic $\langle R, P \rangle$ has an acyclic extension $\langle R', P' \rangle$ in which R' is reflexive.*

The *transitive closure* of a binary relation R is the binary relation R^T defined by $x \, R^T \, y$ if there is a sequence $x_1, x_2, \ldots, x_L$ such that

$$x = x_1 \, R \, x_2 \, R \ldots R \, x_L = y.$$

Equivalently, R^T is the smallest transitive relation containing R, or

$$R^T = \bigcap \{R' : R' \text{ is transitive and } R \subseteq R'\}.$$

Thus, $\langle R, P\rangle$ is acyclic iff there is no x and y with $x\, R^T\, y$ and $y\, P\, x$. Note also that $\langle R, P\rangle$ is acyclic iff $\langle R^T, P\rangle$ is acyclic.

Acyclicity captures a basic property of rational preference. If a binary relation $\succeq$ is complete and $\succ$ is the strict part of $\succeq$, then $\succeq$ is transitive (and hence a rational preference relation) iff $\langle \succeq, \succ\rangle$ is acyclic. As we shall see, acyclicity is essentially what is left of rationality when $\succeq$ is only partially observed, in the sense that we observe some $R \subsetneq \succeq$.

Many problems in revealed preference theory are extension exercises, in the sense that we are given an order pair $\langle R, P\rangle$ and we need to find some well-behaved order pair extension of $\langle R, P\rangle$. The problem then amounts to comparing elements that are left uncompared in R; the following lemma is a basic result on adding comparisons.

Lemma 1.3 (Extension Lemma). *Let $\langle R, P\rangle$ be an acyclic order pair, and $x, y \in X$ be two distinct elements, unordered by R. Then one of the following statements holds true:*

I) $\langle R \cup \{(x, y)\}, P \cup \{(x, y)\}\rangle$ *is acyclic.*
II) $\langle R \cup \{(y, x)\}, P \cup \{(y, x)\}\rangle$ *is acyclic.*
III) $\langle R \cup \{(x, y), (y, x)\}, P\rangle$ *is acyclic.*

Proof. Define a binary relation S by letting $w\, S\, z$ if there is a sequence $x_1, x_2, \ldots, x_L$ such that

$$w = x_1\, R\, x_2\, R \ldots R\, x_L = z,$$

and for which $x_l\, P\, x_{l+1}$ for some $l = 1, \ldots, L-1$. Note that if $x\, S\, y$, then none of the pairs (x_l, x_{l+1}) can equal (x, y) or (y, x) because x and y are unordered by R.

Suppose that (III) is not true: we show that one of the other statements must hold. Because $\langle R, P\rangle$ is acyclic, if $\langle R \cup \{(x, y), (y, x)\}, P\rangle$ has a cycle, either $\langle R \cup \{(x, y)\}, P\rangle$ has a cycle, or $\langle R \cup \{(y, x)\}, P\rangle$ has a cycle. Note that if $\langle R \cup \{(x, y)\}, P\rangle$ has a cycle, then $y\, S\, x$; and if $\langle R \cup \{(y, x)\}, P\rangle$ has a cycle, then $x\, S\, y$. This implies that if (III) is not true then $x\, S\, y$ or $y\, S\, x$.

Suppose that $x\, S\, y$. We show that (I) follows. Since $\langle R, P\rangle$ is acyclic, any cycle of $\langle R \cup \{(x, y)\}, P \cup \{(x, y)\}\rangle$ must involve (x, y). But $x\, S\, y$ implies that a cycle with the same initial and terminal points could be constructed in $\langle R, P\rangle$. So there can be no cycles in $\langle R \cup \{(x, y)\}, P \cup \{(x, y)\}\rangle$.

Similarly, $y\, S\, x$ implies (II).

We present two fundamental results on extending a binary relation. The first is called Szpilrajn's Theorem, and deals with extending a partial order to a linear order. The second is a basic and useful result on extensions of acyclic order pairs to preference relation.

Theorem 1.4 (Szpilrajn). *Suppose that $\succeq$ is a partial order. Then there is a linear order $\succeq'$ which extends $\succeq$.*

Proof. We present a proof for the case when X is finite. The general result relies on Zorn's Lemma: see the proof of Theorem 1.5. We shall use the notation from the proof of Lemma 1.3.

Since $\succeq$ is a partial order, it is antisymmetric. Consider any antisymmetric order R and its strict part P. Suppose x and y are unordered according to R. If we apply Lemma 1.3 to $\langle R,P\rangle$, then in fact (I) or (II) must hold. The reason is that if x and y are distinct elements, and there is a sequence $x_1,x_2,\dots,x_L$ such that

$$x = x_1 \, R \, x_2 \, R \dots R \, x_L = y,$$

then antisymmetry implies that some R may be replaced by P, and hence $x \, S \, y$. As a consequence, if (II) does not hold, there must exist a sequence $x_1,x_2,\dots,x_L$ as above, which means that $x \, S \, y$. In turn, $x \, S \, y$ implies (I), by the argument in the proof of Lemma 1.3.

The proof is completed by induction. We can define a sequence $(\succeq_n)_{n=0}^N$ of binary relations as follows. Let $\succeq_0 = \succeq$. Now suppose we have given $\succeq_{n-1}$ such that $\langle \succeq_{n-1}, \succ_{n-1}\rangle$ is acyclic and $\succeq_{n-1}$ is antisymmetric. Suppose there is a pair x,y unordered according to $\succeq_{n-1}$. Applying Lemma 1.3, either (I) or (II) must hold. Without loss, suppose $\langle \succeq_{n-1} \cup \{(x,y)\}, \succ_{n-1} \cup \{(x,y)\}\rangle$ is acyclic. Define $\succeq_n = \succeq_{n-1} \cup \{(x,y)\}$ and note that $\succ_{n-1} \cup \{(x,y)\}$ is its strict part. Further, $\succeq_n$ retains antisymmetry. Since X is finite, there is N such that $\succeq_N$ is complete, and by construction transitive since $\langle \succeq_N, \succ_N\rangle$ is acyclic.

The following result is perhaps the first "revealed preference theorem" in the book. Its significance will become clear in the next few chapters.

Theorem 1.5 *Let $\langle R,P\rangle$ be an order pair. There is a preference relation $\succeq$ such that $\langle \succeq, \succ\rangle$ is an order pair extension of $\langle R,P\rangle$ iff $\langle R,P\rangle$ is acyclic.*

Proof. It is obvious that if $\succeq$ exists, then $\langle R,P\rangle$ is acyclic. We proceed to prove the converse. Suppose that $\langle R,P\rangle$ is acyclic. By Observation 1.2, we can assume without loss of generality that R is reflexive.

The proof proceeds by first showing that there is a transitive, though not necessarily complete, relation $\hat{R}$ for which $\langle \hat{R}, P_{\hat{R}}\rangle$ is an order pair extension of $\langle R,P\rangle$. Then we use Lemma 1.3 in an induction argument to show that the relation $\hat{R}$ can be completed.

In the first step, define $\hat{R} = (R)^T$, the transitive closure of R. It is obvious that $R \subseteq \hat{R}$. Further, suppose that $x \, P \, y$. Then, by acyclicity of $\langle R,P\rangle$, it follows that $y \, (R)^T \, x$ is impossible, so that $x \, P_{\hat{R}} \, y$.

Now, let $\mathcal{W}$ be the collection of all acyclic order pairs $\langle R^*, P^*\rangle$ that satisfy (a) that $\langle R^*, P^*\rangle$ is an order pair extension of $\langle R,P\rangle$, and (b) that P^* is the strict part of R^*, $P^* = P_{R^*}$. By the first step of the proof, $\mathcal{W} \neq \varnothing$. Order $\mathcal{W}$ by pointwise set inclusion; so that $\langle R^1, P^1\rangle \leq \langle R^2, P^2\rangle$ if $R^1 \subseteq R^2$ and $P^1 \subseteq P^2$. That is, $\langle R^1, P^1\rangle \leq \langle R^2, P^2\rangle$ if $\langle R^2, P^2\rangle$ is an order pair extension of $\langle R^1, P^1\rangle$.

We shall use Zorn's Lemma. Let $\langle R_\lambda, P_\lambda\rangle_{\lambda\in\Lambda}$ be a chain in $\mathcal{W}$. We claim that $\langle\bigcup_{\lambda\in\Lambda} R_\lambda, \bigcup_{\lambda\in\Lambda} P_\lambda\rangle \in \mathcal{W}$. Clearly it is an extension of $\langle R,P\rangle$. It is also clearly acyclic; for, suppose that there is a cycle $x_1 \left(\bigcup_{\lambda\in\Lambda} R_\lambda\right) x_2 \ldots x_n \left(\bigcup_{\lambda\in\Lambda} R_\lambda\right) x_1$, with at least one instance of $\bigcup_{\lambda\in\Lambda} P_\lambda$. Then by definition, there is $\lambda \in \Lambda$ for which $x_1 R_\lambda x_2 \ldots x_n R_\lambda x_1$, with at least one instance of P_λ. Finally, it is clear that $\bigcup_{\lambda\in\Lambda} P_\lambda$ is the strict part of $\bigcup_{\lambda\in\Lambda} R_\lambda$, for if $(x,y) \in \bigcup_{\lambda\in\Lambda} P_\lambda$ and $(y,x) \in \bigcup_{\lambda\in\Lambda} R_\lambda$, there is $\lambda \in \Lambda$ for which $x\, P_\lambda\, y$ and $y\, R_\lambda\, x$, contradicting our hypothesis. And if $(x,y) \in \bigcup_{\lambda\in\Lambda} R_\lambda$ but $(y,x) \notin \bigcup_{\lambda\in\Lambda} R_\lambda$, then there is $\lambda \in \Lambda$ for which $x\, R_\lambda\, y$ but not $y\, R_\lambda\, x$, so that $x\, P_\lambda\, y$.

By Zorn's Lemma, there exists a maximal acyclic order pair $\langle\tilde{R},\tilde{P}\rangle$ for which $\tilde{P}$ is the strict part of $\tilde{R}$ and which extends $\langle R,P\rangle$. Since R is reflexive, so is $\tilde{R}$. We claim that $\tilde{R}$ is complete. If not, then there are $x,y \in X$ with $x \neq y$ which are unranked. By Lemma 1.3, there is an acyclic order pair extension of $\langle\tilde{R},\tilde{P}\rangle$ and satisfying the relevant property, which also clearly extends $\langle R,P\rangle$, contradicting maximality of $\langle\tilde{R},\tilde{P}\rangle$. The desired relation is then $\tilde{R}$.

Theorem 1.5 has a direct application to a family $(\langle R_\lambda, P_\lambda\rangle)_{\lambda\in\Lambda}$ of order pairs.

Corollary 1.6 *There is a preference relation $\succeq$ such that for all $\lambda \in \Lambda$, $\langle\succeq,\succ\rangle$ is an order pair extension of $\langle R_\lambda, P_\lambda\rangle$ iff $\langle\bigcup_{\lambda\in\Lambda} R_\lambda, \bigcup_{\lambda\in\Lambda} P_\lambda\rangle$ is an acyclic order pair.*

We will say that an order pair $\langle R,P\rangle$ is *quasi-acyclic* if there is no sequence $x_1, x_2, \ldots, x_L$ such that

$$x_1\, P\, x_2\, P \ldots P\, x_L,$$

and $x_L\, R\, x_1$.

The following provides a result related to Theorem 1.5, but guaranteeing quasitransitive preference instead of transitive preference.

Lemma 1.7 *There is a complete, quasitransitive relation $\succeq$ such that $\langle\succeq,\succ\rangle$ is an order pair extension of $\langle R,P\rangle$ iff $\langle R,P\rangle$ is quasi-acyclic.*

Proof. As in the proof of Theorem 1.5, one direction is obvious. Suppose instead that $\langle R,P\rangle$ is quasi-acyclic. Define $x \succeq y$ iff it is not the case that $y\, P^T\, x$. We claim that $\succeq$ is the appropriate relation.

We first show that $x \succ y$ iff $x\, P^T\, y$. If $x \succ y$, then since $y \succeq x$ is false, it follows that $x\, P^T\, y$ is true. If, instead, $x\, P^T\, y$ is true, then by acyclicity, $y\, P^T\, x$ is false (this uses the property that $P \subseteq R$). As a consequence, we obtain $x \succeq y$. And since $x\, P^T\, y$, we know that $y \succeq x$ is false, so that $x \succ y$.

Now, note that if $x\, R\, y$, it follows that $y\, P^T\, x$ is false (by quasi-acyclicity), so that $x \succeq y$. Further, if $x\, P\, y$, then by the preceding paragraph, we know that $x \succ y$. This proves that $\langle\succeq,\succ\rangle$ is an extension of $\langle R,P\rangle$.

Further, $\succeq$ is complete. Suppose by means of contradiction that it is not: then there is a pair x,y which are unordered according to $\succeq$. By definition, it must be that $x\, P^T\, y$ and $y\, P^T\, x$. But then we have $x\, P^T\, x$, contradicting quasi-acyclicity (this uses the hypothesis that $P \subseteq R$). And $\succ$ is transitive as $\succ = P^T$, which is by definition transitive.

Finally, we establish one more result. It should be clear by now that, in an order pair $\langle R,P\rangle$, P might not equal P_R, the strict part of R. Starting from Theorem 1.5, however, we have established extensions to pairs in which the second order indeed equals the strict part of the first order. This motivates the question of when such extensions are possible.

Say that an order pair $\langle R,P\rangle$ is *asymmetric* if there are no x,y for which $x\,P\,y$ and $y\,R\,x$. An asymmetric order pair may have cycles, *but it has no cycles of length 2.*

Lemma 1.8 *There is a complete relation $\succeq$ such that $\langle\succeq,\succ\rangle$ is an order pair extension of $\langle R,P\rangle$ iff $\langle R,P\rangle$ is asymmetric.*

Proof. One direction is obvious. For the other direction, define $x \succeq y$ iff it is not the case that $y\,P\,x$. Then it is clear that this relation is complete, as $P \subseteq R$ and $\langle R,P\rangle$ is asymmetric (so that it is impossible that $x\,P\,y$ and $y\,P\,x$). To see that $\langle\succeq,\succ\rangle$ is an extension of $\langle R,P\rangle$: whenever $x\,R\,y$, we have $y\,P\,x$ is false, and consequently that $x \succeq y$. And whenever $x\,P\,y$, then $x\,R\,y$, so that $x \succeq y$. And since $x\,P\,y$, by definition, $(y,x) \notin \succeq$. Thus $x \succ y$.

1.4 CYCLIC MONOTONICITY

Let $X \subseteq \mathbf{R}^n$ be nonempty, and $\rho : X \rightrightarrows \mathbf{R}^n$ be a correspondence. Say that ρ is *cyclically monotone* (or that it satisfies cyclic monotonicity) if, for every finite sequence $x_1,\ldots,x_L$, with $L > 1$ and $x_L = x_1$, and every choice of $z_i \in \rho(x_i)$ for $i = 1,\ldots,L-1$, it holds that

$$\sum_{i=1}^{L-1} z_i \cdot (x_{i+1} - x_i) \geq 0.$$

The property of cyclic monotonicity is important in revealed preference theory. It is a multidimensional generalization of *monotonicity*: suppose that $n = 1$. Then a function $\rho : X \to \mathbf{R}$ is monotone decreasing if $(y-x)$ and $(\rho(y) - \rho(x))$ are of opposite signs; put differently, if

$$0 \leq (y-x)(\rho(x) - \rho(y)) = \rho(x)(y-x) + \rho(y)(x-y).$$

By analogy with the case when $n = 1$, we can say that a correspondence $\rho : X \rightrightarrows \mathbf{R}^n$ is monotone when for all $x,y \in X$ and all $z_x \in \rho(x)$, $z_y \in \rho(y)$,

$$0 \leq z_x \cdot (y-x) + z_y \cdot (x-y).$$

The notion of cyclic monotonicity generalizes the idea of monotonicity to account for all finite cycles in X, not only cycles of length two.

Theorem 1.9 *Suppose that one of the following two hypotheses is satisfied:*

- *For all $x \in X$, $\rho(x) \neq \varnothing$.*
- $|\{x \in X : \rho(x) \neq \varnothing\}| < +\infty$ *and* $0 < \left|\bigcup_{x \in X} \rho(x)\right| < +\infty$.

Then the following statements are equivalent:

I) ρ *is cyclically monotone.*
II) *There is a function* $u : X \to \mathbf{R}$ *such that for all* $x, y \in X$ *and all* $z_x \in \rho(x)$,

$$u(y) - u(x) \le z_x \cdot (y - x).$$

III) *There is a function* $u : X \to \mathbf{R}$ *such that for all* $y \in X$,

$$u(y) = \inf\{u(x) + z_x \cdot (y - x) : x \in X, z_x \in \rho(x)\}.$$

Proof. It is obvious that (III) implies (II). To show that (II) implies (I), note that for any sequence $x_1, \ldots, x_L$, with $x_L = x_1$, it holds that $0 = \sum_{l=1}^{L-1}(u(x_{l+1}) - u(x_l))$.

To prove the theorem, we shall prove that (I) implies the existence of a function u that satisfies the properties stated in (II) and in (III). Let $\rho : X \rightrightarrows \mathbf{R}^n$ be cyclically monotone. Fix an arbitrary $x^* \in X$ for which $\rho(x^*) \neq \varnothing$ (this exists under either of the two hypotheses). Fix $z^* \in \rho(x^*)$.

Let $x \in X$. Here is how we define $u(x)$. Denote by Σ_x the set of all finite sequences $((x_1, z_1), (x_2, z_2), \ldots, (x_{M-1}, z_{M-1}), (x_M))$, with $M \ge 1$, $x_1 = x^*$, $x_M = x$, $z_i \in \rho(x_i)$ and $z_1 = z^*$. Think of sequences $((x_1, z_1), (x_2, z_2), \ldots, (x_{M-1}, z_{M-1}), (x_M))$ as *paths* connecting x^* and $x = x_M$ for which there is $z_i \in \rho(x_i)$ for all x_i, except for x_M. Define $u(x)$ as follows:

$$u(x) = \inf\{\sum_{m=1}^{M-1} z_m \cdot (x_{m+1} - x_m) : ((x_1, z_1), \ldots, (x_M)) \in \Sigma_x\}.$$

The function $u : X \to \mathbf{R}$ is then well defined under either of the two hypotheses. Under the first hypothesis, for any sequence ending in x_M, we may fix $z_M \in \rho(x_M)$ and note that for $((x_1, z_1), \ldots, (x_M))_{m=1}^{M} \in \Sigma_x$,

$$\sum_{m=1}^{M-1} z_m \cdot (x_{m+1} - x_m) + z_M \cdot (x^* - x_M) \ge 0,$$

as ρ is cyclically monotone. Therefore, $z_M \cdot (x - x^*)$ is a lower bound on $\sum_{m=1}^{M-1} z_m \cdot (x_{m+1} - x_m)$ for $((x_1, z_1), \ldots, (x_M))_{m=1}^{M} \in \Sigma_x$, and thus the infimum in the definition of $u(x)$ is defined in $\mathbf{R}$. Under the second hypothesis, it is enough to observe that by cyclic monotonicity we may without loss of generality restrict to the subset of Σ_x which has no repetitions of elements, which by assumption is finite. Hence, the infimum becomes a minimum. Note that, under either case, $u(x^*) = 0$.

We finish the proof by showing that the function u satisfies the properties in (II) and (III). Fix $x, y \in X$ and $z_x \in \rho(x)$. For any $((x_1, z_1), \ldots, (x_M)) \in \Sigma_x$, by definition of $u(y)$,

$$u(y) \le z_x \cdot (y - x) + \sum_{m=1}^{M-1} z_m \cdot (x_{m+1} - x_m);$$

and hence $u(y) - z_x \cdot (y - x)$ is a lower bound on the set in the right hand side of the definition of $u(x)$. Thus $u(y) - z_x \cdot (y - x) \le u(x)$. This establishes (II).

To prove (III), note that (II) implies that $u(y)$ is a lower bound on $\{u(x) + z \cdot (y - x) : x \in X, z \in \rho(x)\}$. Fix $\varepsilon > 0$. By definition of $u(y)$ there is some $((x_1, z_1), \ldots, (x_M)) \in \Sigma_y$ such that $u(y) + \varepsilon > \sum_{m=1}^{M-1} z_m \cdot (x_{m+1} - x_m)$. Let $\hat{x} = x_{M-1}$ and $\hat{z} = z_{M-1}$. Then

$$u(y) + \varepsilon > \hat{z} \cdot (y - \hat{x}) + \sum_{m=1}^{M-2} z_m \cdot (x_{m+1} - x_m) \ge \hat{z} \cdot (y - \hat{x}) + u(\hat{x});$$

which finishes the proof.

When X is convex, the theorem has the important implication that the function u is concave. We record this fact as follows:

Corollary 1.10 *Let $X \subseteq \mathbf{R}^n$ be a convex set. Under either of the hypotheses of Theorem 1.9, the following statements are equivalent:*

I) *ρ is cyclically monotone.*

II) *There is a concave function $u : X \to \mathbf{R}$ such that for all $x, y \in X$ and $z_x \in \rho(x)$*

$$u(y) - u(x) \le z_x \cdot (y - x).$$

III) *There is a concave function $u : X \to \mathbf{R}$ such that for all $y \in X$,*

$$u(y) = \inf\{u(x) + z \cdot (y - x) : x \in X, z \in \rho(x)\}.$$

Proof. The definition $u(y) = \inf\{u(x) + z \cdot (y - x) : x \in X, z \in \rho(x)\}$ in statement (III) implies that u is concave, as it is the lower envelope of affine functions. The result follows because the function constructed in the proof of statements (II) and (III) of Theorem 1.9 is the same.

Versions of these results exist even without either of the hypotheses of Theorem 1.9. However, one must allow for the possibility that u is no longer real-valued.

The importance of (II) in Corollary 1.10 should be clear from the following result.

Proposition 1.11 *Let $X \subseteq \mathbf{R}^n$ be convex, and $u : X \to \mathbf{R}$ be a concave function. Then, for all $x \in$ int (X) there is $p \in \mathbf{R}^n$ such that*

$$u(y) \le u(x) + p \cdot (y - x),$$

for all $y \in X$.

The proof of Proposition 1.11 is an application of the separating hyperplane theorem, and is omitted.

For any function $u : X \to \mathbf{R}$ and $x \in X$, if $p \in \mathbf{R}^n$ has the property that $u(y) \le u(x) + p \cdot (y - x)$, for all $y \in X$, then p is a *supergradient* of u at x. Proposition 1.11 says that if u is concave then it has a supergradient at any interior point of its domain. The set of all supergradients of u at x is called the *superdifferential* of u at x.

When u is concave and differentiable at x the only supergradient at x is the gradient of u at x. So the superdifferential of u at x is $\rho(x) = \{\nabla u(x)\}$.

1.5 THEOREM OF THE ALTERNATIVE

Many results in revealed preference theory can be formulated using some version of the Theorem of the Alternative, or Farkas' Lemma. We state a version (Lemma 1.12) that may involve either real or rational coefficients and solutions. The usefulness of such a result is that it allows one to state a problem involving real solutions to a system of linear inequalities, and find a revealed preference axiom from the rational solution to the alternative linear system.

For a matrix B, we denote by B_i its ith row.

Lemma 1.12 *Let* $\mathbf{F}$ *be either the set of real numbers* $\mathbf{R}$*, or the set of rational numbers* $\mathbf{Q}$*. Let* B *be a* $(M_1+M_2)\times K$ *matrix with entries in* $\mathbf{F}$*. Consider the systems of inequalities S1 and S2:*

$$S1: \begin{cases} B_i \cdot \theta \geq 0, i = 1,\ldots,M_1 \\ B_i \cdot \theta > 0, i = M_1+1,\ldots,M_1+M_2 \end{cases}$$

$$S2: \begin{cases} \eta \cdot B = 0 \\ \eta \geq 0, \end{cases}$$

where θ *is the* K*-dimensional unknown in S1 and* η*, of dimension* (M_1+M_2)*, is the unknown in S2. Then S1 has a solution* $\theta \in \mathbf{F}^K$ *iff S2 has no solution* $\eta \in \mathbf{F}^{M_1+M_2}$ *with* $\eta_i > 0$ *for some* $i \in \{M_1+1,\ldots,M_1+M_2\}$.

Lemma 1.13 *(Integer–Real Farkas) Let* $\{A_i\}_{i=1}^M$ *be a finite collection of vectors in* $\mathbf{Q}^K$*. Then one and only one of the following statements is true:*

I) *There exists* $y \in \mathbf{R}^K$ *such that for all* $i = 1,...,L$, $A_i \cdot y \geq 0$ *and for all* $i = L+1,...,M$, $A_i \cdot y > 0$.

II) *There exists* $z \in \mathbf{Z}_+^M$ *such that* $\sum_{i=1}^M z_i A_i = 0$*, where* $\sum_{i=L+1}^M z_i > 0$.

Proof. Both (I) and (II) cannot simultaneously hold. To see why, suppose that there exist y and z as stated in (I) and (II). Then $A_i \cdot y \geq 0$ for all $i = 1,\ldots,L$ and $A_i \cdot y > 0$ for all $i = L+1,\ldots,M$. Consider $\sum_{i=1}^M z_i A_i \cdot y$. Since $\sum_{i=1}^M z_i A_i = 0$, we know that $\sum_{i=1}^M z_i A_i \cdot y = 0$. Furthermore, since there is some $j \in \{L+1,\ldots,M\}$ for which $z_j A_j \cdot y > 0$, and for all i, $z_i A_i \cdot y \geq 0$, we conclude that $\sum_{i=1}^M z_i A_i \cdot y > 0$, a contradiction.

We now establish that if (II) does not hold, (I) holds. Note that (II) holds iff there is $z \in \mathbf{Q}_+^M$ satisfying the statement in (II). (The reason being that M is finite and that z satisfies the statement in (II) iff Nz satisfies it, for any positive integer N.) By Lemma 1.12 if (II) does not hold, there exists $y \in \mathbf{Q}^K$ such that for all $i = 1,...,L, A_i \cdot y \geq 0$ and for all $i = L+1,...,M, A_i \cdot q > 0$. Since $\mathbf{Q}^K \subseteq \mathbf{R}^K$, $y \in \mathbf{R}^K$.

Finally, the following nonhomogeneous version, which is easily proved using Lemma 1.12, often comes in useful.

Lemma 1.14 *Let B be an $M \times K$ matrix with real entries and let γ be an M-vector with real entries. Consider the systems of inequalities S1 and S2:*

$$S1 : B_i \cdot \theta \geq \gamma_i, i = 1, \ldots, M$$

$$S2 : \begin{cases} \eta \cdot B = 0 \\ \eta \geq 0 \\ \eta \cdot \gamma > 0 \end{cases}$$

where θ is the K-dimensional unknown in S1 and η, of dimension M, is the unknown in S2. Then S1 has a solution $\theta \in \mathbf{R}^K$ iff S2 has no solution $\eta \in \mathbf{R}^M$.

1.6 CHAPTER REFERENCES

Szpilrajn's Theorem (Theorem 1.4) is due to Szpilrajn (1930). Theorem 1.5 generalizes Szpilrajn's Theorem, and is due to Richter (1966) and Hansson (1968). It appears in the form stated here in Suzumura (1976b). Sen (1969) popularized the notion of quasitransitivity. The original proofs of Theorem 1.1 were based on a construction due to Cantor (1895), which shows that any two countable dense linearly ordered sets without endpoints are order-isomorphic, and first appears in economics in Debreu (1954).

Theorem 1.9 and Corollary 1.10 are standard results in convex analysis: see Rockafellar (1997). The idea of cyclic monotonicity and the fact that it characterizes subsets of superdifferentials first appears in Rockafellar (1966). Rockafellar also observes that monotonicity is equivalent to cyclic monotonicity for $X = \mathbf{R}$. The condition of cyclic monotonicity appears frequently in the mechanism design literature as well. Rochet (1987) is a direct generalization of Rockafellar's result. Since monotonicity is necessary and sufficient for cyclic monotonicity in one dimension, one might conjecture that eliminating cycles of length n is necessary and sufficient for cyclic monotonicity in $n-1$ dimensions. This turns out to be false. An example appears in Asplund (1970) (a mapping which rotates every $x \in \mathbf{R}^2$ by $\frac{\pi}{n}$ is cyclically monotonic of order n but not $n+1$). See also Bartz, Bauschke, Borwein, Reich, and Wang (2007). However, there are characterizations of monotonic correspondences which naturally generalize the Rockafellar result; see Krauss (1985) or Fitzpatrick (1988).

Theorems of the Alternative, such as Lemma 1.12, make an early appearance in Farkas (1902); classic references in economics are Kuhn and Tucker (1956) and Gale (1960b). The version here for the real numbers appears first in

Motzkin (1936), and can be derived from the more general formulation due to Slater (1951). The rational version is essentially Theorem 3.2 of Fishburn (1973b). A general result along these lines is Theorem 1.6.1 in Stoer and Witzgall (1970). A more recent general reference is Schrijver (1998). Lemma 1.14 can be found, for example, in Gale (1960b), and Lemma 1.13 in Chambers and Echenique (2014b).