CHAPTER 2

Classical Abstract Choice Theory

We start our development of revealed preference theory by discussing the abstract model of choice. All revealed preference problems have two components: *data*, and *theory*. Given a family of possible data, and a particular theory, a revealed preference exercise seeks to describe the particular instances of data that are compatible with the theory. We shall illustrate the role of each component for the case of abstract choice. The data consist of observed choices made by an economic agent. A theory describes a criterion, or a mechanism, for making choices.

Given is a set X of objects that can possibly be chosen. In principle, X can be anything; we do not place any structure on X. A collection of subsets $\Sigma \subseteq 2^X \backslash \{\varnothing\}$ is given, called the *budget* sets. Budget sets are potential sets of elements from which an economic agent might choose. A *choice function* is a mapping $c : \Sigma \rightarrow 2^X \backslash \{\varnothing\}$ such that for all $B \in \Sigma$, $c(B) \subseteq B$. Importantly, choice from each budget is nonempty.

For the present chapter, choice functions are going to be our notion of data. The interpretation of a choice function c is that we have access to the choices made by an individual agent when facing different sets of feasible alternatives. A particular choice function, then, embodies multiple observations.

The main theory is that of the maximization of some binary relation on X. The theory postulates that the agent makes choices that are "better" than other feasible choices, where the notion of better is captured by a binary relation. The theory will be refined by imposing assumptions on the binary relation: for example that the relation is a preference relation (i.e. a weak order).

Given notions of data and theory, the problem is to understand when the former are consistent with the latter. We are mainly going to explore two ways of formulating this notion of consistency.

We say that a binary relation $\succeq$ on X *strongly rationalizes* c if for all $B \in \Sigma$,

$$c(B) = \{x \in B : \forall y \in B, x \succeq y\}.$$

When there is such a binary relation, we say that c is *strongly rationalizable*.

The idea behind strong rationalizability is that $c(B)$ should comprise *all* of the best elements of B according to $\succeq$.

In contrast, we may only want to require that $c(B)$ be among the best elements in B: Say that a binary relation $\succeq$ on X *weakly rationalizes* choice function c if for all $B \in \Sigma$,

$$c(B) \subseteq \{x \in B : \forall y \in B, x \succeq y\}.$$

A utility function u weakly rationalizes the choice function c if the binary relation $\succeq$ defined by $x \succeq y \Leftrightarrow u(x) \geq u(y)$ weakly rationalizes c.

Weak rationalizability allows for the existence of feasible alternatives that are equally as good as the chosen ones, but that were not observed to be chosen.

It should be clear from the definitions that all choice functions are weakly rationalizable by the binary relation $X \times X$ – meaning the binary relation defined by $x \succeq y$ for all x and y (or, in other words, that $x \sim y$ for all $x, y \in X$). For the exercise to be interesting, one must impose constraints on the rationalizing $\succeq$: such constraints can be thought of as a *discipline* on the revealed preference exercise. The need to impose various kinds of discipline shall emerge more than once in this book.

In contrast with weak rationalizability, strong rationalizability does rule out some choice functions: There are choice functions that are not strongly rationalizable. We say that strong rationalizability is *testable*, or that it has *observable implications*. Our first results, in Section 2.1 to follow, seek to describe precisely those choice functions that are strongly rationalizable, and to discuss rationalization by binary relations with particular properties.

2.1 STRONG RATIONALIZATION

Given a choice function c, we can define its *revealed preference pair* $\langle \succeq^c, \succ^c \rangle$ by $x \succeq^c y$ iff there exists $B \in \Sigma$ such that $\{x,y\} \subseteq B$ and $x \in c(B)$, and $x \succ^c y$ iff there exists $B \in \Sigma$ such that $\{x,y\} \subseteq B$, $x \in c(B)$, and $y \notin c(B)$. The binary relations in $\langle \succeq^c, \succ^c \rangle$ give rise to the name "revealed preference theory." The idea, of course, is that if an agent's choices are guided by a preference relation, then $x \in c(B)$ and $y \in B$ when x is at least as good as y according to the agent's preferences. Thus $x \succeq^c y$ captures those binary comparisons which are revealed by c to be part of the agent's preferences.

It is important to recognize that, in general, $\succ^c$ need not be asymmetric, and it need not be the strict part of $\succeq^c$. In general, though, $\succ^c \subseteq \succeq^c$. Thus, it is an order pair according to our definition. In fact, the notion of an order pair is meant to capture precisely the pairs of orders arising in revealed preference theory.

The theory developed here is based on the revealed preference pair. It is, however, important to caution that there may be relevant information in a choice function that is not contained in the revealed preference pair. The following example presents two choice functions that give rise to the same revealed preference pair. The first is strongly rationalizable (by a quasitransitive relation), while the other is not.

Example 2.1 *Let $X = \{x,y,z,w\}$. Consider $\Sigma = \{\{x,y,z,w\},\{y,z,w\}\}$. Define a choice function $c(\{x,y,z,w\}) = \{x,y\}$ and $c(\{y,z,w\}) = \{y,z\}$. This choice function is rationalizable by the reflexive binary relation which ranks $x \sim y$, $x \succ z$, $y \sim z$ and each of x, y, and z above w. This relation is quasitransitive.*

Now, let $\Sigma' = \{\{x,y\},\{y,z\},\{x,z\},\{x,w\},\{y,z,w\}\}$ where $c'(\{x,y\}) = \{x,y\}$, $c'(\{y,z\}) = \{y\}$, $c'(\{x,z\}) = \{x\}$, $c'(\{x,w\}) = \{x\}$, and $c'(\{y,z,w\}) = \{y,z\}$. Note this generates exactly the same revealed preference pair as the preceding, but c' is not rationalizable by any relation: The reason is that a rationalizing relation would have to have $y \succ z$ (from $c'(\{y,z\}) = \{y\}$) and $y \sim z$ (from $c'(\{y,z,w\}) = \{y,z\}$).

We begin with a description of strongly rationalizable choice functions. The following observation, a direct consequence of the definition, almost gives us our first answer.

Proposition 2.2 *A choice function c is strongly rationalizable iff it is strongly rationalizable by $\succeq^c$.*

Proof. Suppose c is strongly rationalizable. Then there exists $\succeq$ which strongly rationalizes c. Let $B \in \Sigma$, and let $x \in c(B)$. By definition, for all $y \in B$, $x \succeq^c y$. On the other hand, suppose that $x \succeq^c y$ for all $y \in B$. For any $y \in B$, since $x \succeq^c y$, there is $B_y \in \Sigma$ for which $y \in B_y$ and $x \in c(B_y)$. Since $\succeq$ strongly rationalizes c, we conclude that $x \succeq y$ and thus that for all $y \in B$, $x \succeq y$. Then $x \in c(B)$ as $\succeq$ strongly rationalizes c.

Given Proposition 2.2, it is easy to formulate a condition that says that $\succeq^c$ rationalizes c. The condition is called the V-*axiom*: A choice function c satisfies the V-axiom if for all $B \in \Sigma$ and all $x \in B$, if $x \succeq^c y$ for all $y \in B$, then $x \in c(B)$.

Theorem 2.3 *A choice function is strongly rationalizable iff it satisfies the V-axiom.*

Proof. By Proposition 2.2, we need to show that the V-axiom is equivalent to strong rationalizability by $\succeq^c$. But that $x \in c(B)$ and $y \in B$ implies $x \succeq^c y$ is just the definition of $\succeq^c$, so strong rationalizability by $\succeq^c$ is the statement that $x \in c(B)$ iff for all $y \in B$, $x \succeq^c y$, which is just the V-axiom.

Many revealed preference exercises boil down to an extension exercise; in which, given a revealed preference pair, we seek an order pair extension with particular properties. Recall that when we write the order pair $\langle \succeq, \succ \rangle$, then $\succ$ is the strict part of $\succeq$. This may not be the case for $\langle \succeq^c, \succ^c \rangle$.

Theorem 2.4 *A binary relation $\succeq$ strongly rationalizes c if $\langle \succeq, \succ \rangle$ is an order pair extension of $\langle \succeq^c, \succ^c \rangle$.*

Proof. Suppose that $\langle \succeq, \succ \rangle$ is an order pair extension of $\langle \succeq^c, \succ^c \rangle$. We need to show that for all $B \in \Sigma$, $x \in c(B)$ if and only if $x \succeq y$ for all $y \in B$. So suppose that $x \in c(B)$ and $y \in B$. Then by definition, $x \succeq^c y$. Consequently, $x \succeq y$. Now suppose that $x \succeq y$ for all $y \in B$. Since $c(B) \neq \varnothing$, if $x \notin c(B)$ then there is $y \in B$ for which $y \succ^c x$. Then $y \succ x$, as $\langle \succeq, \succ \rangle$ is an order pair extension of $\langle \succeq^c, \succ^c \rangle$, which would contradict that $x \succeq y$ for all $y \in B$. So $x \in c(B)$.

A converse to Theorem 2.4 is available when $\succeq$ is a preference relation:

Theorem 2.5 *A preference relation $\succeq$ strongly rationalizes c iff $\langle \succeq, \succ \rangle$ is an order pair extension of $\langle \succeq^c, \succ^c \rangle$.*

Proof. Given Theorem 2.4, we shall only prove the necessity of order pair extension.

Suppose that c is strongly rationalizable by some preference relation $\succeq$. The definition of strong rationalization implies that if $x \succeq^c y$, then $x \succeq y$. Suppose that $x \succ^c y$. Then there is B for which $\{x, y\} \subseteq B$, $x \in c(B)$ and $y \notin c(B)$. Since $y \notin c(B)$, completeness of $\succeq$ implies that there is $z \in B$ with $z \succ y$. Since $x \in c(B)$, we know $x \succeq z$. By transitivity of $\succeq$, $x \succ y$.

We now turn to strong rationalization by a preference relation (a weak order). The condition should be familiar from our discussion in Chapter 1. Say that a choice function c is *congruent* if $\langle \succeq^c, \succ^c \rangle$ is acyclic. Our next result is the fundamental characterization of rationalization by a preference relation; it follows quite directly from the results we have established in Chapter 1.

Theorem 2.6 *A choice function is strongly rationalizable by a preference relation iff it is congruent.*

Proof. By Theorem 2.5, there is a preference relation $\succeq$ rationalizing c iff there is a preference relation $\succeq$ for which $\langle \succeq, \succ \rangle$ is an order pair extension of $\langle \succeq^c, \succ^c \rangle$. By Theorem 1.5, this is true iff $\langle \succeq^c, \succ^c \rangle$ is acyclic.

It is frequently convenient to work with single-valued choice functions, choice functions such that for every $B \in \Sigma$, $c(B)$ is a singleton. The following result says that many natural assumptions one could want to place on such choice functions are equivalent: one could say *observationally equivalent*, in the sense that they strongly rationalize the same choice functions.

Proposition 2.7 *Suppose for all $B \in \Sigma$, B is finite, and $|c(B)| = 1$. Then the following statements are equivalent:*

I) *c is strongly rationalizable by a complete, quasitransitive relation.*
II) *c is strongly rationalizable by a preference relation.*
III) *c is strongly rationalizable by a strict preference relation.*
IV) *$\succ^c$ is acyclic.*

Proof. That (III) implies (II) and (II) implies (I) are obvious, the latter because a transitive relation is always quasitransitive. We show that (I) implies (III). First, note that $\succeq^c = (\succ^c \cup =)$, because c is single-valued. We know that there is a complete and quasitransitive $\succeq$ which rationalizes c. We will show that $\succ^c$

$\subseteq \succ$. So suppose that $x \succ^c y$. This means that there is $B \in \Sigma$ for which $\{x,y\} \subseteq B$, $x \in c(B)$ and $y \notin c(B)$. Because $y \notin c(B)$ and $\succeq$ is complete, there exists $y_2 \in B$ for which $y_2 \succ y_1 = y$. If $y_2 \notin c(B)$, there is $y_3 \in B$ for which $y_3 \succ y_2$. We can inductively continue this construction, and because $\succeq$ is quasitransitive, $\succ$ is acyclic, and B is finite, there is y_k such that $y_k \succ y_{k-1} \succ \dots y_1 = y$, where $y_k \in c(B)$. By quasitransitivity, $y_k \succ y$. But since $c(B)$ is single-valued, $y_k = x$, so that $x \succ y$, which is what we wanted to show.

Now, we let $\succeq^*$ be a strict preference relation for which $x \succ y$ implies $x \succ^* y$, which exists by Theorem 1.4 (note that $(\succ \cup =)$ is a partial order by quasitransitivity of $\succeq$). We claim that $\succeq^*$ rationalizes c. So let $x \in c(B)$, which means that for all $y \in B$ where $y \neq x$, $x \succ^c y$, so that $x \succ y$ and finally $x \succ^* y$. Conversely, suppose that $x \succeq^* y$ for all $y \in B$. Then it must be that $x \succeq y$ for all $y \in B$, as otherwise, there would exist $y \in B$ for which $y \succ x$, which would imply $y \succ^* x$. Consequently, as $\succeq$ rationalizes c, we have $x \in c(B)$.

Finally, (I) is equivalent to (IV). It is easy to see that if c is single-valued and quasitransitive rationalizable, then $\succ^c$ is acyclic. Conversely, suppose that $\succ^c$ is acyclic. Since c is single-valued, $\succeq^c = (\succ^c \cup =)$. By Lemma 1.7, there is a quasitransitive $\succeq$ for which $\succeq^c \subseteq \succeq$ and $\succ^c \subseteq \succ$. The result now follows by Theorem 2.4.

2.1.1 Weak axiom of revealed preference

Congruence in Theorem 2.6 requires one to rule out cycles of any length (using the terminology of Chapter 1). A weaker condition only rules out cycles of length two. We say that a choice function satisfies the *weak axiom of revealed preference* if whenever $x \succeq^c y$, it is not the case that $y \succ^c x$.[1]

While the weak axiom of revealed preference has less bite than congruence, its weakness can be compensated for by a condition on Σ, the domain of the choice function. Note that the larger is the collection Σ, the more restrictive is the weak axiom. So one can make up for the weakness of the weak axiom by demanding a large collection Σ. In particular, if Σ includes all sets of cardinality at most three, the weak axiom of revealed preference is equivalent to rationalizability by a preference relation.

The following pair of results gives the implications of the weak axiom for choice on arbitrary domains. We then present a result which gives the implications of the weak axiom when Σ is rich, in a sense to be made precise.

Theorem 2.8 *A choice function c satisfies the weak axiom of revealed preference iff there exists a complete binary relation $\succeq$ which strongly rationalizes c such that $\succ$ extends $\succ^c$* (i.e. $\succ^c \subseteq \succ$).

[1] The weak axiom of revealed preference is an instance of the condition we termed asymmetry of an order pair in Chapter 1. Indeed, by Lemma 1.8, the weak axiom of revealed preference is the weakest hypothesis that establishes the existence of a complete relation $\succeq$ such that $\langle \succeq, \succ \rangle$ is a order pair extension of $\langle \succeq^c, \succ^c \rangle$.

Proof. First, let $\succeq$ be a complete rationalizing relation such that $\succ$ extends $\succ^c$. Since $\succeq$ strongly rationalizes c, $\succeq^c \subseteq \succeq$. Therefore, if $x \succeq^c y$ and $y \succ^c x$, we would have $x \succeq y$ and $y \succ x$, a contradiction with the definition of $\succ$.

On the other hand, suppose c satisfies the weak axiom of revealed preference. By Lemma 1.8, there is a complete $\succeq$ for which $\succeq^c \subseteq \succeq$ and $\succ^c \subseteq \succ$. By Theorem 2.4, $\succeq$ rationalizes c.

Theorem 2.9 *Suppose that Σ contains all sets of cardinality at most three. Then c is strongly rationalizable by a preference relation iff it satisfies the weak axiom of revealed preference.*

Proof. That a choice function rationalizable by a preference relation satisfies the weak axiom of revealed preference is obvious, as the weak axiom is implied by congruence. Conversely, suppose that c satisfies the weak axiom of revealed preference. By assumption, for all $x, y \in X$, $\{x, y\} \in \Sigma$. Since $c(\{x, y\}) \neq \varnothing$, this implies that $\succeq^c$ is complete. Next, we claim that it is transitive. For suppose that $x \succeq^c y$ and $y \succeq^c z$. If either $x = y$, $y = z$, or $x = z$, then it is obvious that $x \succeq^c z$. So suppose they are all distinct. To prove $x \succeq^c z$, we will show that $x \in c(\{x, y, z\})$, for if not, either $y \in c(\{x, y, z\})$, in which case $y \succ^c x$ (contradicting the weak axiom); or $y \notin c(\{x, y, z\})$ and $z \in c(\{x, y, z\})$, in which case $z \succ^c y$, again contradicting the weak axiom. Thus $x \in c(\{x, y, z\})$, so that $x \succeq^c z$, and $\succeq^c$ is transitive. This shows that $\succeq^c$ is a preference relation.

Finally, $\succ^c$ is the strict part of $\succeq^c$. Clearly, $x \succ^c y$ implies $x \succeq^c y$, and, by the weak axiom, also implies that $y \succeq^c x$ is false. Conversely, if $x \succeq^c y$ and $y \succeq^c x$ is false, we know that $x = c(\{x, y\})$, so that $x \succ^c y$. By Theorem 2.4, c is strongly rationalizable by $\succeq^c$.

Note that, under the hypotheses of Theorem 2.9, the revealed preference relation $\succeq^c$ strongly rationalizes c (Proposition 2.2). By observing choice from all sets of cardinality two, one can identify uniquely the rationalizing preference. For all x and y, $\{x, y\} \in \Sigma$, so $c(\{x, y\})$ is defined and nonempty. So x and y must be ordered by $\succeq^c$.

There are two conditions that are well known to "decompose" the weak axiom of revealed preference. We give them the names first used by Sen (1969), and we show that they are equivalent to the weak axiom.

Say that c satisfies *condition α* if, for any $B, B' \in \Sigma$ where $B \subseteq B'$, $c(B') \cap B \subseteq c(B)$. Say that c satisfies *condition β* if for all $B, B' \in \Sigma$ where $B \subseteq B'$, if $\{x, y\} \subseteq c(B)$, then $x \in c(B')$ if and only if $y \in c(B')$.

The weak axiom refers implicitly to choice from two sets: the set at which x is chosen over y when we say that $x \succeq^c y$, and the set at which y is chosen over x when we say that $y \succ^c x$. Conditions α and β talk explicitly about these sets; α constrains choice from a smaller set, given what was chosen at a larger set, while condition β goes in the opposite direction. The following result explains that α and β can be said to decompose the weak axiom.

Theorem 2.10 *If a choice function satisfies the weak axiom, then it satisfies conditions α and β. Conversely, suppose that either of the two following properties is satisfied:*

I) *Σ has the property that whenever $B, B' \in \Sigma$ have nonempty intersection, then $B \cap B' \in \Sigma$*
II) *For all x, y, $\{x, y\} \in \Sigma$.*

Then if a choice function satisfies conditions α and β, it satisfies the weak axiom.

Proof. Let c be a choice function with domain Σ. Suppose that c satisfies the weak axiom. We shall prove that it satisfies α and β. Let $B, B' \in \Sigma$ with $B \subseteq B'$.

Consider first condition α. Suppose that $x \in c(B') \cap B$. Then $x \succeq^c y$ for all $y \in B'$. In particular, $x \succeq^c y$ for any $y \in c(B)$, as $B \subseteq B'$. If $x \notin c(B)$, then there is $z \in c(B)$ (which is assumed nonempty), so that $z \succ^c x$, violating the weak axiom. Hence $x \in c(B)$.

Second, consider condition β. Let $\{x, y\} \subseteq c(B)$ and $B \subseteq B'$. Then $x \succeq^c y$ and $y \succeq^c x$. Suppose $x \in c(B')$. If $y \notin c(B')$, we have $y \succ^c x$, contradicting the weak axiom. Hence $y \in c(B')$. A symmetric argument establishes that if $y \in c(B')$, then $x \in c(B')$.

Conversely, suppose c satisfies conditions α and β. Suppose, toward a contradiction, that there are x and y with $x \succeq^c y$ and $y \succ^c x$. There are then $A, A' \in \Sigma$ with $\{x, y\} \subseteq A \cap A'$, $x \in c(A)$, $y \in c(A')$ and $x \notin c(A')$. Let $B = A \cap A'$ if condition (I) is satisfied, and let $B = \{x, y\}$ if condition (II) is satisfied. Then by condition α, $x \in c(A)$ and $y \in c(A')$ imply that $x, y \in c(B)$. Now condition β implies that we cannot have $y \in c(A')$ and $x \notin c(A')$. Hence c satisfies the weak axiom.

Corollary 2.11 *Suppose that Σ contains all sets of cardinality at most three. Then c is strongly rationalizable by a preference relation iff it satisfies conditions α and β.*

Proof. Follows from Theorem 2.9 and Theorem 2.10.

When choice $c(B)$ is a singleton for all B, then condition β has no bite. Our next result considers choice functions with singleton values. It explains why some authors reserve the phrase “weak axiom of revealed preference” for what we have called condition α.

Theorem 2.12 *Suppose that Σ has the property that whenever $B, B' \in \Sigma$ have nonempty intersection, then $B \cap B' \in \Sigma$. Suppose further that for all $B \in \Sigma$, $|c(B)| = 1$. Then c satisfies the weak axiom of revealed preference iff it satisfies condition α.*

Proof. Note that condition β is vacuous under the single-valuedness hypothesis. The result therefore follows directly from Theorem 2.10.

2.1.2 Maximal rationalizability

Strong rationalizability requires that $c(B)$ consist of the "best" alternatives in B for a binary relation. Instead, one can require that $c(B)$ contain all the elements of B that cannot be "improved" upon. If the binary relation in question is complete, the two approaches will be equivalent. In general, however, they differ.

Say that a binary relation $\succeq$ *maximally rationalizes* c if

$$c(B) = \{x \in B : \nexists y \in B, y \succ x\}.$$

That is, $c(B)$ corresponds to a set of undominated elements.

If there exists a binary relation $\succeq$ that maximally rationalizes c, then there exists a binary relation $\succeq'$ which strongly rationalizes c. Namely, $x \succeq' y$ if $y \succ x$ is false.[2] When $\succeq$ is complete, it is easy to see that $\succeq' = \succeq$.

The following example shows that there are choice functions that are strongly rationalizable but not maximal rationalizable.

Example 2.13 *Let $\Sigma = \{\{x,y\},\{x,z\},\{x,y,z\}\}$, and $c(\{x,y\}) = \{x,y\}$, $c(\{x,z\}) = \{x,z\}$, and $c(\{x,y,z\}) = \{x\}$. Then this choice function is strongly rationalizable, but is not maximal rationalizable. The relation $\succeq$ that strongly rationalizes c is given by $x \succeq y$, $y \succeq x$, $x \succeq z$, $z \succeq x$, $x \succeq x$, $y \succeq y$, and $z \succeq z$. Note that $\succeq$ is not complete.*

To see that c is not maximal rationalizable, suppose that $\succeq$ maximally rationalizes c. Then because $c(\{x,y,z\}) = \{x\}$, it must be that either $x \succ y$ or $z \succ y$. But $x \succ y$ is impossible, as $c(\{x,y\}) = \{x,y\}$. On the other hand, suppose that $z \succ y$. Then it follows that $x \succ z$, or else $z \in c(\{x,y,z\})$. But this contradicts $z \in c(\{x,z\})$.

2.1.3 Quasitransitivity

The previous analysis makes use of the property of transitivity of a preference relation. In contrast, a quasitransitive relation presents a challenge. In particular, a version of Theorem 2.6 for strong rationalization by a quasitransitive relation is not available (see also Theorem 11.2). But one can easily prove a similar result in one direction:

Proposition 2.14 *If $\langle \succeq^c, \succ^c \rangle$ is quasi-acyclic, then there is a complete and quasitransitive relation strongly rationalizing c.*

Proof. Suppose $\langle \succeq^c, \succ^c \rangle$ is quasi-acyclic. By Lemma 1.7, there is a complete and quasitransitive relation $\succeq$ for which $\succeq^c \subseteq \succeq$ and $\succ^c \subseteq \succ$. By Theorem 2.4, $\succeq$ strongly rationalizes c.

But quasi-acyclicity of $\langle \succeq^c, \succ^c \rangle$ is not necessary for strong rationalization by a quasitransitive relation, as shown by our next example.

[2] $\succeq'$ is then termed the *canonical conjugate* of $\succ$ (Kim and Richter, 1986). This object appears often in revealed preference theory.

Example 2.15 *Suppose* $X = \{x,y,z\}$*, and let* $\Sigma = \{\{x,y\},\{y,z\},\{x,z\},\{x,y,z\}\}$*. Define* $c(\{x,y\}) = \{x,y\}$*,* $c(\{y,z\}) = \{y,z\}$*,* $c(\{x,z\}) = \{x\}$*, and* $c(\{x,y,z\}) = \{x,y\}$*. Then c is strongly rationalizable by the relation* $\succeq$ *given by*

$$\succeq = \{(x,y),(y,x),(y,z),(z,y),(x,z),(x,x),(y,y),(z,z)\};$$

that is, all pairs are indifferent except for x and z. The relation $\succeq$ *is quasitransitive. However,* $z \succeq^c y$ *and* $y \succ^c z$ *so that* $\langle \succeq^c, \succ^c \rangle$ *is not quasi-acyclic.*

Proposition 2.7 presents a characterization of quasitransitive strong rationalization in the case when c is singleton-valued. In general, however, it is difficult to find a simple characterization of quasitransitive rationalizability. The reasoning is simple. When we have a preference relation $\succeq$ strongly rationalizing c, we know that if $x \in c(B)$ and $y \in B$, but $y \notin c(B)$, then $x \succ y$. The same cannot be said to hold for quasitransitive relations. In fact, suppose that $\succeq$ is a complete relation on a finite set X for which $\succ$ is acyclic. A relatively easy proposition states that this $\succeq$ is a preference relation iff for all finite B, if $y \in B$ but $y \notin c(B)$, then *for all* $x \in c(B)$, $x \succ y$. On the other hand, it can be shown that $\succeq$ is quasitransitive iff for all finite sets B, if $y \in B$ and $y \notin c(B)$, then *there exists* $x \in c(B)$, $x \succ y$.

2.2 SATISFICING

The notion of strong rationalization can be adapted to the study of a theory of "bounded rationality," the theory of satisficing behavior. The theory, which was first proposed by Herbert Simon, claims that an individual always makes choices which are "good enough," though not necessarily optimal. In our context, the assumption is that if an individual chooses an alternative, then she also chooses anything which is at least as good.

We will say that a choice function c is *satisficing rationalizable* by $\succeq$ if for all $B \in \Sigma$ and all $x,y \in B$, $x \in c(B)$ and $y \succeq x$ imply $y \in c(B)$.

The following two results characterize various notions of satisficing rationalizability. They are essentially extension results, whereby instead of searching for $\succeq$ for which $\succeq^c \subseteq \succeq$ and $\succ^c \subseteq \succ$, we simply require $\succ^c \subseteq \succ$.

Theorem 2.16 *The following statements are equivalent:*

I) *c is satisficing rationalizable by a strict preference relation.*
II) *c satisficing rationalizable by a preference relation.*
III) $\succ^c$ *is acyclic.*

Proof. First, note that if c is satisficing rationalizable by a strict preference relation, then it is obviously satisficing rationalizable by a preference relation.

Second, we prove that (II) implies (III). So suppose that c is satisficing rationalizable by a preference relation $\succeq$, but suppose that there is a cycle $x_1 \succ^c x_2 \ldots \succ^c x_K \succ^c x_1$. We claim that $x_i \succ^c x_{i+1}$ implies that $x_i \succ x_{i+1}$. Note that as $x_i \succ^c x_{i+1}$, there exists $B \in \Sigma$ for which $\{x_i, x_{i+1}\} \subseteq B$, $x_i \in c(B)$, and

$x_{i+1} \notin c(B)$. Because $\succeq$ satisficing rationalizes c, it follows that $x_i \succ x_{i+1}$; as otherwise, we must have $x_{i+1} \in c(B)$. This demonstrates that there is a $\succ$ cycle, contradicting the fact that $\succeq$ is a preference relation. This shows that (II) implies (III).

Finally, we show that (III) implies (I). So suppose that $\succ^c$ admits no cycles. We can consider the transitive closure of the relation $\{(x,y) : x \succ^c y$ or $x = y\}$; denote it by $\succeq'$. By acyclicity of $\succ^c$, $\succeq'$ is a partial order. By Szpilrajn's Theorem (Theorem 1.4), it admits an extension to a strict preference relation $\succeq$. We claim that $\succeq$ satisficing rationalizes c. To see this, suppose that $B \in \Sigma$, $x \in c(B)$, and $y \succeq x$. Then it must be the case that $y \in c(B)$, as otherwise, we would have $x \succ^c y$, which would imply $x \succ y$, a contradiction.

We say a choice function satisfies the *weakened weak axiom of revealed preference* if $\succ^c$ is asymmetric. The following theorem states that the weakened weak axiom is necessary and sufficient for the existence of a complete binary relation $\succeq$ for which $\succ^c \subseteq \succ$.

Theorem 2.17 *A choice function is satisficing rationalizable by a complete binary relation iff it satisfies the weakened weak axiom of revealed preference.*

Proof. First, suppose that c is satisificing rationalizable by a complete binary relation $\succeq$, and suppose by way of contradiction that there exist x and y for which $x \succ^c y$ and $y \succ^c x$. Then there exist $B, B' \in \Sigma$ for which $\{x,y\} \subseteq B \cap B'$, $x \in c(B)$, $y \notin c(B)$, $y \in c(B')$, $x \notin c(B')$. As $\succeq$ satisficing rationalizes c, we therefore conclude that $x \succ y$ and $y \succ x$, a contradiction.

On the other hand, suppose that c satisfies the weakened weak axiom of revealed preference. We define $x \succeq y$ when $y \succ^c x$ is false. Note that by the weakened weak axiom of revealed preference, $\succeq$ is complete. Further, $\succeq$ satisficing rationalizes c. To see this, suppose that $\{x,y\} \in B$ for some $B \in \Sigma$, and that $x \in c(B)$ and $y \succeq x$. Then if $y \notin c(B)$, we know that $x \succ^c y$. This implies that $y \succ^c x$ is false (by the weakened weak axiom), so that $x \succeq y$. And since $x \succ^c y$, we know that $y \succeq x$ is false (by definition of $\succeq$). We conclude that $x \succ y$, a contradiction.

Note that there is an obvious parallel between Theorem 2.6 and Theorem 2.16 on the one hand, and between Theorem 2.8 and Theorem 2.17 on the other. However, there is no counterpart of Theorem 2.9 here. For example, a choice function c defined on all finite subsets of $\{x,y,z\}$ by $c(\{x,y\}) = \{x\}$, $c(\{y,z\}) = \{y\}$, $c(\{x,z\}) = \{z\}$, $c(\{x,y,z\}) = \{x,y,z\}$ satisfies the weakened weak axiom of revealed preference, but there is no transitive relation $\succeq$ that satisficing rationalizes c.

2.3 WEAK RATIONALIZATION

The notion of strong rationalization is based on the assumption that $c(B)$ could capture all the choices that an agent might potentially make from a set B. In general, though, in an empirical environment, we have no reason to expect that we will be able to observe such an object. What we usually observe is a choice that an agent makes; but this does not preclude the possibility of other choices being reasonable for that agent. The notion of weak rationalization is intended to capture this idea.

The following theorem is an analogue to Theorem 2.5. However, it does not require the hypothesis that $\succeq$ be a preference relation. Thus, in general, the property of being weakly rationalizable by a relation $\succeq$ is completely characterized by the revealed preference pair.

Theorem 2.18 *A binary relation $\succeq$ weakly rationalizes c iff the revealed preference pair $\langle \succeq^c, \succ^c \rangle$ satisfies $\succeq^c \subseteq \succeq$.*

Proof. Suppose c is weakly rationalizable by some $\succeq$. Note that if $x \succeq^c y$, then by definition, $x \succeq y$. So $\succeq^c \subseteq \succeq$.

On the other hand, suppose that the revealed preference pair $\langle \succeq^c, \succ^c \rangle$ satisfies $\succeq^c \subseteq \succeq$. Then we claim that $\succeq$ weakly rationalizes c. We need to show that for all $B \in \Sigma$, $x \in c(B)$ implies $x \succeq y$ for all $y \in B$. So suppose that $x \in c(B)$. Then by definition, $x \succeq^c y$. Consequently, $x \succeq y$.

Weak rationalization is trivial if we place no restrictions on the order which can weakly rationalize a choice function. This is because complete indifference weakly rationalizes anything. To this end, we will study rationalization by preference relations which have *monotonicity* properties. Monotonicity will be a discipline imposed on the revealed preference exercise.

Let $\langle \geq, > \rangle$ be an acyclic order pair. Say that a binary relation $\succeq$ is *monotonic* with respect to $\langle \geq, > \rangle$ if $\langle \succeq, \succ \rangle$ is an order pair extension of $\langle \geq, > \rangle$. The meaning of $\langle \geq, > \rangle$ is that $\geq$ and $>$ reflect some observable characteristics of the alternatives under consideration, and that we can require the unobservable rationalizing relations to somehow conform to $\geq$ and $>$.

The order pair $\langle \geq, > \rangle$ suggests a structure on budget sets. Say that $B \subseteq X$ is *comprehensive* with respect to order pair $\langle \geq, > \rangle$ if whenever $x \in B$ and $x \geq y$, $y \in B$. The notion of a comprehensive budget is natural when budgets are defined to be a set of objects that are "affordable." Suppose that B is defined by some notion of what the agent can purchase, like a budget defined from prices and income (see Chapter 3). Then, if $x \in B$ because the cost of purchasing x is below the agent's income, and $x \geq y$ implies that the cost of purchasing y cannot exceed that of purchasing x, then of course we have that $y \in B$.

For a fixed $\langle \geq, > \rangle$, define the order pair $\langle \succeq^R, \succ^R \rangle$ by $x \succeq^R y$ if $x \succeq^c y$ and define $x \succ^R y$ if there exists $B \in \Sigma$ and $z \in B$ where $\{x, y, z\} \subseteq B$, $x \in c(B)$ and $z > y$.

The following theorem shows that the order pair $\langle \succeq^R, \succ^R \rangle$ is the proper tool for studying rationalization by a preference relation which is monotonic with respect to $\langle \geq, > \rangle$. We say that a choice function satisfies the *generalized axiom of revealed preference* if the order pair $\langle \succeq^R, \succ^R \rangle$ is acyclic. Theorem 2.19 is relevant to the issues we shall study in Chapter 3.

Theorem 2.19 *Suppose that the acyclic order pair $\langle \geq, > \rangle$ satisfies $x > y \geq z$ implies $x > z$, and that all $B \in \Sigma$ are comprehensive. Then there exists a preference relation which is monotonic with respect to order pair $\langle \geq, > \rangle$ and which weakly rationalizes c iff $\langle \succeq^R, \succ^R \rangle$ satisfies the generalized axiom of revealed preference. In addition, if there is a countable set $Y = \{y_1, y_2, \ldots\}$ such that for all x, z satisfying $x > z$, there is k such that $x > y_k > z$, then the preference relation can be chosen to have a utility representation.*

Proof. Suppose preference relation $\succeq$ is monotonic with respect to $\langle \geq, > \rangle$ and weakly rationalizes c. Suppose by way of contradiction that $\langle \succeq^R, \succ^R \rangle$ is not an acyclic order pair. If it is not acyclic, there is $x_1, \ldots, x_L$ for which $x_1 \succeq^R x_2 \ldots \succeq^R x_L$ and $x_L \succ^R x_1$. Since $x_L \succ^R x_1$, there is z for which $x_L \succeq^R z$ and $z > x_1$. Since $\succeq$ weakly rationalizes c, we have $x_1 \succeq \ldots \succeq x_L \succeq z$, and by monotonicity, we have $z \succ x_1$, a contradiction to the fact that $\succeq$ is a preference relation.

Conversely, suppose that $\langle \succeq^R, \succ^R \rangle$ is an acyclic order pair. We shall demonstrate that $\langle \succeq^R \cup \geq, > \rangle$ is also an acyclic order pair.[3] It will therefore follow by Theorem 1.5 that there is a preference relation $\succeq$ for which $\succeq^R \subseteq \succeq$, $\geq \subseteq \succeq$, and $> \subseteq \succ$. As a consequence, $\succeq$ will be monotonic with respect to $\langle \geq, > \rangle$, and by Theorem 2.18, $\succeq$ will weakly rationalize c (since $\succeq^R = \succeq^c$). So, suppose for a contradiction that $\langle \succeq^R \cup \geq, > \rangle$ is not an acyclic order pair. The key observations here are that for all $x, y, z \in X$:

- $x \succeq^R y \geq z$ implies $x \succeq^R z$
- $x \succeq^R y > z$ implies $x \succ^R z$
- $x \succ^R y \geq z$ implies $x \succ^R z$

Indeed, the first implication follows because if $x \succeq^R y$, there exists $B \in \Sigma$ for which $\{x, y\} \subseteq B$ and $x \in c(B)$. But if $y \in B$, by comprehensivity, $z \in B$ as well, so that $x \succeq^R z$. The second implication follows by definition. The third implication follows since, if $x \succ^R y$, then there exist $B \in \Sigma$ and $w \in B$ for which $x \in c(B)$ and $w > y$. As B is comprehensive and $y \geq z$, $z \in B$. Further, as $w > y \geq z$, we have by assumption that $w > z$. Consequently $x \succ^R z$.

Since $\langle \succeq^R \cup \geq, > \rangle$ is not acyclic, there is a $\langle \succeq^R \cup \geq, > \rangle$-cycle. For simplicity, let $Q = \left(\succeq^R \cup \geq\right)$. Let $x_1 \, Q \ldots Q \, x_L > x_1$ be a $\langle \succeq^R \cup \geq, > \rangle$-cycle, where $L \geq 2$. By the preceding observations, we can, without loss of generality, assume that this cycle takes the form

$$x_1 \geq x_2 \geq \ldots \geq x_K \succeq^R \ldots \succeq^R x_L > x_1 \tag{2.1}$$

by converting all relations of the form $x_i \succeq^R x_{i+1} \geq x_{i+2}$ to $x_i \succeq^R x_{i+2}$.

If $K \neq L$, then we have $x_{L-1} \succeq^R x_L > x_1$. Here, the second observation implies that $x_{L-1} \succ^R x_1$. But $x_1 \geq \ldots \geq x_K$, so by repeatedly applying the third observation, we would have $x_{L-1} \succ^R x_K$. But then $x_K \succeq^R \ldots \succeq^R x_{L-1} \succ^R x_K$, contradicting acyclicity of $\langle \succeq^R, \succ^R \rangle$.

[3] For future reference, note that this also implies that $\langle \succeq^R \cup \geq, \succ^R \cup > \rangle$ is an acyclic order pair.

On the other hand, if $K = L$, then the cycle in equation (2.1) is a $\langle \geq, > \rangle$ cycle, contradicting the acyclicity of $\langle \geq, > \rangle$. We have therefore established that $\langle \succeq^R \cup \geq, > \rangle$ is acyclic.

To see the statement about the utility representation, let $S = \left(\succeq^R \cup \geq\right)$, and let $T = \left(\succ^R \cup >\right)$. Define the relation U by $x \, U \, y$ if there are $x_1, \ldots, x_k$ with $x = x_1 \, V \ldots V \, x_k = y$, where each $V \in \{S, T\}$, and at least one instance coincides with T. Note that U is transitive. Define $u(x) = \sum_{\{k : x U y_k\}} 2^{-k}$.

First, we claim that u represents $\succeq$ which weakly rationalizes c. Thus, suppose that $x \in c(B)$, and that $y \in B$. Now, if $y \, U \, y_k$, then we have $x \succeq^R y \, U \, y_k$, whereby $x \, U \, y_k$, so that $u(x) \geq u(y)$. Second, we claim that $\succeq$ is also monotonic with respect to $\langle \geq, > \rangle$. Thus, suppose $x \geq y$. Again, if $y \, U \, y_k$, then by definition $x \geq y \, U \, y_k$, so that $x \, U \, y_k$. Hence $u(x) \geq u(y)$. Now, suppose that $x > y$. Then there is y_k for which $x > y_k > y$. We claim that $y \, U \, y_k$ is false (obviously $x \, U \, y_k$). So, suppose by way of contradiction that $y \, U \, y_k$ holds, so that $y \, U \, y_k > y$. This implies the existence of a $\langle \succeq^R \cup \geq, > \rangle$ cycle, which we previously demonstrated to be impossible.

If one is willing to sacrifice the transitivity of indifference, nearly any choice function becomes weakly rationalizable by a monotonic binary relation. As we shall see, the relevant property of c is that for all x, $x \succ^R x$ is false. In fact, any choice function satisfying this property can be rationalized by the relation defined as $x \succeq y$ if $y > x$ is false. Let us denote this relation as $>^{-1}$.

Proposition 2.20 *Suppose $\langle \geq, > \rangle$ is an acyclic order pair for which $x > y \geq z$ implies $x > z$. Then $>^{-1}$ is complete, quasitransitive, and monotonic with respect to $\langle \geq, > \rangle$.*

Proof. To show that $>^{-1}$ is complete, suppose that it is not. Then there are x, y for which $x > y$ and $y > x$. But this contradicts the acyclicity of order pair $\langle \geq, > \rangle$ (as $> \subseteq \geq$). Further, $>^{-1}$ is quasitransitive. Suppose $x >^{-1} y$ and $y >^{-1} z$, but that $y >^{-1} x$ and $z >^{-1} y$ are false. It follows that $x > y$ and $y > z$ must be true. As a consequence of our assumption that $x > y \geq z$ implies $x > z$, this implies that $x > z$, which implies by acyclicity that $x >^{-1} z$, and of course $z >^{-1} x$ is false. It is clear by acyclicity that $>^{-1}$ is monotonic with respect to $\langle \geq, > \rangle$.

Theorem 2.21 *Suppose that the acyclic order pair $\langle \geq, > \rangle$ satisfies $x > y \geq z$ implies $x > z$, and that all $B \in \Sigma$ are comprehensive. Then the following statements are equivalent:*

I) *There exists a complete binary relation which is monotonic with respect to order pair $\langle \geq, > \rangle$ and which weakly rationalizes c.*
II) *For all x, $x \succ^R x$ is false.*
III) *$>^{-1}$ weakly rationalizes c.*

Proof. First, we show that (I) implies (II). So suppose that there is a complete order monotonic with respect to order pair $\langle \geq, > \rangle$ and weakly rationalizing c.

We claim that for all x, $x \succ^R x$ is false. Suppose by way of contradiction that there exists x for which $x \succ^R x$. Then by definition there exists $B \in \Sigma$ and $y \in B$ for which $x \in c(B)$ and $y > x$. But since $y > x$ and $\succeq$ is monotonic with respect to order pair $\langle \geq, > \rangle$, it follows that $y \succ x$. This implies that $x \notin c(B)$, as $\succeq$ weakly rationalizes c, a contradiction.

To see that (II) implies (III), we need to show that $>^{-1}$ weakly rationalizes c; suppose that $x \in c(B)$. Suppose by way of contradiction that there is $y \in B$ such that $y >^{-1} x$ and $x >^{-1} y$ is false. Then it follows that $y > x$. But since $x \in c(B)$, $y \in B$, and $y > x$, it follows that $x \succ^R x$, a contradiction.

Finally, that (III) implies (I) is obvious.

The theorem demonstrates that the only empirical content of quasitransitive choice is that for all B, if $x \in c(B)$, then x lies on the "boundary" of B. The equivalence of (I) and (III) establishes that, amongst binary relations monotonic with respect to $\langle \geq, > \rangle$, the assumption of quasitransitivity adds no additional empirical content to the hypothesis of weak rationalization by a complete binary relation. This stands in strong contrast to weak rationalization by a preference relation.

2.4 SUBRATIONALIZABILITY

A dual notion to the notion of weak rationalizability is what we will call subrationalizability. We will say that a choice function c is *subrationalizable* by $\succeq$ if for all $B \in \Sigma$,

$$\varnothing \neq \{x \in B : x \succeq y \text{ for all } y \in B\} \subseteq c(B).$$

Thus, a subrationalizable choice function admits all dominant elements of $\succeq$, but may admit other alternatives. We might be interested in such a condition in an environment where an individual potentially makes mistakes.

Fishburn introduces the following notions. For a finite collection $\Sigma' \subseteq \Sigma$, we define

$$\mathcal{C}(\Sigma') = \left\{ x \in \bigcup_{B \in \Sigma'} B : \text{For all } B \in \Sigma', x \in B \Rightarrow x \in c(B) \right\}.$$

That is, $\mathcal{C}(\Sigma')$ is the set of all alternatives that are always chosen from Σ' when they are available. We will say that choice function c satisfies *partial congruence* if for all $\Sigma' \subseteq \Sigma$ where $0 < |\Sigma'| < +\infty$, $\mathcal{C}(\Sigma') \neq \varnothing$.

As its name suggests, partial congruence is weaker than congruence.

Proposition 2.22 *If c satisfies congruence, then it satisfies partial congruence.*

Proof. Suppose c violates partial congruence, and let $\Sigma' \subseteq \Sigma$ for which $0 < |\Sigma'| < +\infty$ and $\mathcal{C}(\Sigma') = \varnothing$. Now, for each $x \in \bigcup_{B \in \Sigma'} c(B)$, there is $y \in \bigcup_{B \in \Sigma'} c(B)$ for which $y \succ^c x$. Pick an arbitrary $x \in \bigcup_{B \in \Sigma'} c(B)$, and label this x_1. For each x_i, find $x_{i+1} \in \bigcup_{B\in\Sigma'} c(B)$ for which $x_{i+1} \succ^c x_i$. Since $|\Sigma'| < \infty$, eventually there will be $i > j$ for which there is $B \in \Sigma'$ so that $\{x_i, x_j\} \subseteq c(B)$. Observe that $x_i \succ^c \ldots \succ^c x_j$, and $x_j \succeq^c x_i$, contradicting congruence.

The following result characterizes subrationalizability by preference relations.

Theorem 2.23 *Suppose a choice function has the property that $|c(B)| < +\infty$ for all $B \in \Sigma$. Then the following are equivalent:*

I) *c is subrationalizable by a preference relation.*
II) *c is subrationalizable by a strict preference relation.*
III) *c satisfies partial congruence.*

The requirement that $|c(B)| < +\infty$ for all $B \in \Sigma$ is necessary to complete a critical induction step in the proof. The general characterization of subrationalizable choice functions remains open. The condition of partial congruence may be problematic from a falsifiability perspective. The reason is that if $\bigcup_{E\in\Sigma'} E$ is an infinite set, the statement that there exists $x \in \mathcal{C}(\Sigma')$ is existential. This means that it postulates the existence of a certain object. If our process of scientific observation consists of observing chosen elements of choice sets one by one, and if we suppose that we can only observe finite data, we can never falsify the hypothesis that such an element exists.

Proof. To see that (I) implies (II), suppose that c is subrationalizable by a preference relation. One simply needs to "break ties." To see this, suppose that $\succeq$ is a preference relation subrationalizing c. Let $\geq$ be a well-ordering of X.[4] We define $\succeq'$ by $x \succeq' y$ if $x \succ y$ or $x \sim y$ and $y \geq x$. First, for every $B \in \Sigma$, there exists a $\succeq'$-maximal element. To see this, consider the collection of $\succeq$-maximal elements, and pick the $\geq$-minimal element (this necessarily exists because $\geq$ is a well-ordering). Then this element is clearly $\succeq'$-maximal. And further, every $\succeq'$-maximal element of B is also $\succeq$-maximal. So $\succeq'$ subrationalizes c, and is a strict preference relation.

To see that (II) implies (III), suppose c is subrationalizable by a strict preference relation $\succeq$. We need to show that it satisfies partial congruence. To this end, let $\Sigma' \subseteq \Sigma$ be a finite subcollection of budgets. Then for each $B \in \Sigma'$, there exists a unique maximal element $x(B) \in B$ according to $\succeq$. By letting x^* be the maximal element of $\{x(B) : B \in \Sigma'\}$ (this exists as Σ' is a finite collection), we see that $x^* \in \mathcal{C}(\Sigma')$.

To see that (III) implies (I), suppose that c satisfies partial congruence. We add to Σ all singleton and binary sets which are not originally present, and extend c to these sets by $c(\{x,y\}) = \{x,y\}$. This new choice function is also clearly partially congruent, and we will show that it is subrationalizable (this

[4] A *well-ordering* of a set X is a strict preference relation for which every nonempty subset of X has a minimal element. Existence of well-orderings is guaranteed by the axiom of choice.

will establish that c is subrationalizable). So, without loss, we can assume that Σ includes all binary sets.

Now, let $\mathcal{E}$ be the collection of finite subsets of Σ. For a partial order $\succeq$ and $\Sigma' \in \mathcal{E}$, we define $\mathcal{C}(\Sigma',\succeq) = \{x \in \bigcup_{E\in\Sigma'} E : \nexists y \in \bigcup_{E\in\Sigma'} E, y \succ x\}$. That is, $\mathcal{C}(\Sigma',\succeq)$ are the $\succeq$-maximal elements of $\bigcup_{E\in\Sigma'} E$.

We consider the collection $\mathcal{R}$ of all partial orders $\succeq$ on X such that for all $\Sigma' \in \mathcal{E}$, $\mathcal{C}(\Sigma',\succeq) \cap \mathcal{C}(\Sigma') \neq \varnothing$. We know that $\mathcal{R}$ is nonempty, as the relation $\Delta = \{(x,x) : x \in X\}$ satisfies $\Delta \in \mathcal{R}$. We will show that $\mathcal{R}$ contains at least one strict preference relation $\succeq$; this will be enough to complete the proof. To see why, note that if $\Sigma' \in \mathcal{E}$ is $\Sigma' = \{B\}$, $\mathcal{C}(\Sigma',\succeq)$ is just the $\succeq$-maximal element of B. And $\mathcal{C}(\Sigma') = c(B)$, so that we conclude that the $\succeq$-maximal element of B is an element of $c(B)$.

So, we order $\mathcal{R}$ by set inclusion. For any chain $\{\succeq_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{R}$, we claim that $\succeq' = \left(\bigcup_{\lambda\in\Lambda} \succeq_\lambda\right) \in \mathcal{R}$. We will thus be able to use Zorn's Lemma to establish the existence of a maximal element. First, it is easy to verify that $\succeq'$ is a partial order; we will skip the details. We need to show that $\succeq'$ satisfies $\mathcal{C}(\Sigma',\succeq') \cap \mathcal{C}(\Sigma') \neq \varnothing$ for all $\Sigma' \in \mathcal{E}$. For a contradiction, suppose that there exists some $\Sigma' \in \mathcal{E}$ for which $\mathcal{C}(\Sigma',\succeq') \cap \mathcal{C}(\Sigma') = \varnothing$. In particular, this implies that for every $x \in \mathcal{C}(\Sigma')$, there exists $y \in \bigcup_{B\in\Sigma'} B$ for which $y \succ' x$. Now, note that as $c(E)$ is finite for each $E \in \Sigma$, this necessarily implies that $\mathcal{C}(\Sigma')$ is also finite (since Σ' is finite). This in particular implies that there exists $\succeq_\lambda$ such that for each $x \in \mathcal{C}(\Sigma')$, there exists $y \in \bigcup_{B\in\Sigma'} B$ for which $y \succ_\lambda x$. But this contradicts the fact that $\succeq_\lambda \in \mathcal{R}$. We conclude by Zorn's Lemma that there exists a maximal $\succeq^* \in \mathcal{R}$. Our only task now is to show that $\succeq^*$ is complete.

Now, suppose conversely that $\succeq^*$ is not complete. Then there exist $x', y' \in X$ for which x' and y' remain unranked according to $\succeq^*$. Recall from Chapter 1 that ^{T} denotes transitive closure. We consider two possible extensions of $\succeq^*$: $\succeq^1 = \left(\succeq^* \cup \{(x',y')\}\right)^T$, and $\succeq^2 = \left(\succeq^* \cup \{(y',x')\}\right)^T$. The two differ in how they rank x' and y'. It is easy to verify that $x \succeq^1 y$ if $x \succeq^* y$ or $x \succeq^* x'$ and $y' \succeq^* y$, and that $x \succeq^2 y$ if $x \succeq^* y$ or $x \succeq^* y'$ and $x' \succeq^* y$.

Because $\succeq^*$ is maximal and each of $\succeq^1$ and $\succeq^2$ are partial orders (antisymmetry can be proved by supposing it to be false and establishing a contradiction that x' and y' were ranked according to $\succeq^*$), we conclude that there exist $\Sigma^1, \Sigma^2 \in \mathcal{E}$ for which $\mathcal{C}(\Sigma^1) \cap \mathcal{C}(\Sigma^1,\succeq^1) = \varnothing$ and $\mathcal{C}(\Sigma^2) \cap \mathcal{C}(\Sigma^2, \succeq^2) = \varnothing$.

This implies that

$$w \in \mathcal{C}(\Sigma^1) \cap \mathcal{C}(\Sigma^1,\succeq^*) \Rightarrow \exists x \in \bigcup_{E\in\Sigma^1} E \text{ such that } x \succeq^* x' \text{ and } y' \succeq^* w, \tag{2.2}$$

and that

$$z \in \mathcal{C}(\Sigma^2) \cap \mathcal{C}(\Sigma^2,\succeq^*) \Rightarrow \exists x \in \bigcup_{E\in\Sigma^2} E \text{ such that } x \succeq^* y' \text{ and } x' \succeq^* z. \tag{2.3}$$

Finally, let $\Sigma^* = \Sigma^1 \cup \Sigma^2 \cup \{\{x',y'\}\}$. Since Σ includes all singleton and binary sets, $\Sigma^* \in \mathcal{E}$, so that $\mathcal{C}(\Sigma^*) \cap \mathcal{C}(\Sigma^*,\succeq^*) \neq \varnothing$. So pick $b \in \mathcal{C}(\Sigma^*) \cap$

$\mathcal{C}(\Sigma^*,\succeq^*)$. We claim that $b \in \{x',y'\}$. Observe that

$$\mathcal{C}(\Sigma^*)\cap\mathcal{C}(\Sigma^*,\succeq^*) \subseteq \left[\mathcal{C}(\Sigma^1)\cap\mathcal{C}(\Sigma^1,\succeq^*)\right]\cup\left[\mathcal{C}(\Sigma^2)\cap\mathcal{C}(\Sigma^2,\succeq^*)\right]\cup\{x',y'\}.$$

So, suppose that $b \in \mathcal{C}(\Sigma^1)\cap\mathcal{C}(\Sigma^1,\succeq^*)$. By (2.2), we conclude that $b = y'$—otherwise, (2.2) tells us that $y' \succ^* b$ (recall $\succeq^*$ is a partial order), so that $b \notin \mathcal{C}(\Sigma^*)\cap\mathcal{C}(\Sigma^*,\succeq^*)$. Analogously, by (2.3), if $b \in \mathcal{C}(\Sigma^2)\cap\mathcal{C}(\Sigma^2,\succeq^*)$, we conclude that $b = x'$.

This establishes that $b \in \{x',y'\}$. So, suppose without loss of generality that $b = x'$. Since $\mathcal{C}(\Sigma^1)\cap\mathcal{C}(\Sigma^1,\succeq^*) \neq \varnothing$, we can conclude by (2.2) that there exists $z \in \bigcup_{E\in\Sigma^1} E$ such that $z \succeq^* x'$. But unless we have $z = x'$, it would follow that $x' \notin \mathcal{C}(\Sigma^*,\succeq^*)$, which we know to be false. So we must conclude that $x' = z \in \bigcup_{E\in\Sigma^1} E$. But by definition of $\mathcal{C}$, this implies that $x' \in \mathcal{C}(\Sigma^1)$, as $x' \in \mathcal{C}(\Sigma^*)$. Finally, $x' \in \mathcal{C}(\Sigma^*,\succeq^*)$ implies $x' \in \mathcal{C}(\Sigma^1,\succeq^1)$, as $z \succ^1 x'$ implies $z \succ^* x'$.

So we have shown that $\mathcal{C}(\Sigma^1)\cap\mathcal{C}(\Sigma^1,\succeq^1) \neq \varnothing$, a contradiction. Similarly, if we assume that $b = y'$, we arrive at the conclusion that $\mathcal{C}(\Sigma^1)\cap\mathcal{C}(\Sigma^2,\succeq^2) \neq \varnothing$. As either possibility arrives at a contradiction, we have established that $\succeq^*$ is complete, and hence a strict preference relation.

2.5 EXPERIMENTAL ELICITATION OF CHOICE

Until now, we have abstained from assigning any concrete interpretation to the notion of a choice function. However, for choice to be an empirical concept, it must be observable. There are conceptual and practical issues that arise when we operationalize the idea that choice should be observable.

Imagine a choice-theoretic experiment, the goal of which is to study a particular subject's choice function. In the experiment, the subject may be presented with multiple budgets. If the goal of the experiment is to understand what the individual would choose from each of the budgets, we run into a basic problem. Imagine an individual presented with choices from the two budgets: $\{\text{hat}, \text{left shoe}\}$, $\{\text{jacket}, \text{right shoe}\}$. Suppose that, when presented with the budget $\{\text{hat}, \text{left shoe}\}$, the choice would be the hat, and when presented with the choice $\{\text{jacket}, \text{right shoe}\}$, the choice would be the jacket. If presented with both choices simultaneously, it is possible that the subject might choose the pair of shoes. The two shoes are complements. The presence of complementarities distorts choice.

There is a classical solution to this problem. The idea is to ask the subject to announce which choice she would make from each budget, and then to randomly select a budget, paying the subject the announced choice from that budget. Suppose Σ is finite. Consider a finite set of states of the world Ω, to be realized in the future, and associate each $B \in \Sigma$ with some state $\omega_B \in \Omega$. If the subject is asked to report a single-valued choice function, and announces c, the subject is paid the random variable paying off the single element of $c(B)$ in state ω_B, which we can also call c with a slight abuse of notation. This mechanism is referred to as the *random decision selection* mechanism.

Note that under the random decision selection mechanism, the subject's ultimate payoff is an element of X^{Ω}, rather than of X. Thus, what matters for her choices is her preference over X^{Ω}, rather than her preference over X. We would expect there to be some relation between preferences on X and preferences on X^{Ω}. We can say a preference $\succeq^*$ over X^{Ω} is *monotonic* with respect to a preference $\succeq$ over X if whenever $f, g \in X^{\Omega}$ satisfy $f(\omega) \succeq g(\omega)$ for all $\omega \in \Omega$, then $f \succeq^* g$, with a strict preference if in addition there is $\omega \in \Omega$ for which $f(\omega) \succ g(\omega)$.[5] Monotonicity is usually understood as the claim that, if the subject is made better off *ex-post*, regardless of which state obtains, then the subject is made better off *ex-ante*.[6]

The punchline is that if $\succeq$ is a complete and transitive relation over X, and $\succeq^*$ over X^{Ω} is monotonic with respect to $\succeq$, then a single-valued choice function c $\succeq^*$ dominates all elements of X^{Ω} iff for all $B \in \Sigma$, if $c(B) = \{y\}$, then $y \succeq x$ for all $x \in B$. Thus, if we are willing to assume monotonicity, then the random decision selection mechanism is a good way of eliciting choice. Importantly, monotonicity requires no form of separability (such as Savage's P2 "sure thing" principle), or an expected utility hypothesis.

2.6 CHAPTER REFERENCES

Probably one of the first works to mention abstract choice theory is Arrow (1951). The theory is mostly a generalization of the classical demand context, described in the next section. Uzawa (1956) is one of the first papers to study the revealed preference approach in an abstract environment, followed shortly thereafter by Arrow (1959), to whom Theorem 2.9 is due. Corollary 2.11 and Theorem 2.12 are due to Sen (1971), while condition α first appears in Chernoff (1954). Theorem 2.6 is first established in full generality in the independent works of Richter (1966) and Hansson (1968). The main contribution in those papers was to describe restrictions on budget sets; prior to this, most works in abstract choice assumed that every finite set could be a potential budget set.

The term "congruence" is due to Richter. Theorem 2.3 on rationalizability is due to Richter (1971). A related result, characterizing maximal rationalizability, is provided by Bossert, Sprumont, and Suzumura (2005). Theorem 2.8 on the class of choice functions satisfying the weak axiom of revealed preference is from Wilson (1970), see also Mariotti (2008). Wilson notes the connection between choice functions satisfying the weak axiom and those induced as von Neumann–Morgenstern stable sets. See also Plott (1974). The characterizations of satisficing rationalizability by preference relation

[5] This assumes no state is null, so that we believe each state might possibly occur; see Chapter 8.

[6] This statement is compelling so long as the *ex-post* notion of "better off" coincides with the *ex-ante* notion. That is, in evaluating *ex-post* outcomes, the *ex-ante* preference $\succeq$ over certain prospects is applied. There are classical examples (Diamond, 1967 or Machina, 1989) where we would not expect this relationship to hold.

and strict preference relations appear in Aleskerov, Bouyssou, and Monjardet (2007) and Tyson (2008). The notion of the weakened weak axiom of revealed preference is due to Ehlers and Sprumont (2008), Theorem 2.17 appears there, and the concept in that characterization appears in Wilson (1970) as the notion of a "Q cut." Theorem 2.23 is due to Fishburn (1976).

Theorem 2.19 is related to Afriat's Theorem, discussed formally in the next chapter. A result along the lines of Theorem 2.19 appears in Quah, Nishimura, and Ok (2013).

The theory of satisficing behavior in Section 2.2 was proposed by Simon (1955). The observations on the distinction between rationalizability and dominant rationalizability are due to Suzumura (1976a). Kim (1987) studies generalized transitivity concepts which we have not discussed, all of which turn out to be empirically equivalent to the standard concept. Bossert and Suzumura (2010) is a detailed work devoted to studying the empirical content of many generalized choice models, some of which we describe here.

A recent literature has used choice as a primitive to study various "behavioral" theories. Manzini and Mariotti (2007) consider choice by the successive application of different binary relations. Masatlioglu, Nakajima, and Ozbay (2012) study attention, and formalize the notion that a decision maker may only consider a subset of the available alternatives. Ok, Ortoleva, and Riella (2014) develop a model of reference-dependent choice. Green and Hojman (2007), de Clippel and Eliaz (2012), Cherepanov, Feddersen, and Sandroni (2013), and Ambrus and Rozen (2014) describe models of agents with multiple motivations. Finally, we should mention the paper by de Clippel and Rozen (2012), which studies some of these behavioral developments under the assumption of limited observability that is very relevant to the notion of empirical content.

In a similar spirit, Green and Hojman (2007), Bernheim and Rangel (2007), Chambers and Hayashi (2012), and Bernheim and Rangel (2009) use choice theory to propose welfare criteria that remain valid when standard revealed preference axioms fail.

The random decision selection mechanism is originally due to Allais (1953), and discussed also by Savage (1954) and his "hot man" example. The idea is also used in the famous elicitation mechanism of Becker, DeGroot, and Marschak (1964). The framework described here is taken from Azrieli, Chambers, and Healy (2012).