CHAPTER 3

Rational Demand

Revealed preference theory started out as an exploration into the testable implications of neoclassical demand theory, and while it has expanded in many different directions, the analysis of rational demand is the most actively researched area in revealed preference theory. In this chapter, we present an exposition of the basic results in the revealed preference theory of rational demand.

We suppose here that we have observations on the purchasing decisions of a single consumer. The consumer makes a sequence of independent choices at different price vectors. The data consists of the consumer's choices, and we seek to understand the implications of rational consumption behavior for such data.

The material on rational demand is divided into three chapters. In Chapter 3 we discuss the basic results on weak and strong rationalization, including Afriat's Theorem, the main result in the revealed preference theory of rational demand. In Chapter 4 we turn to specific properties of demand functions; and in Chapter 5 to some of the practical issues that arise when applying the results of revealed preference theory to empirical research.

3.1 WEAK RATIONALIZATION

Consider an agent choosing a bundle of n goods to purchase. Consumption space is $X \subseteq \mathbf{R}^n_+$, meaning that the consumer chooses $x \in X$. We assume that for any $x \in X$ and $\varepsilon > 0$, there is ε' with $0 < \varepsilon' < \varepsilon$ and $x + \varepsilon'\mathbf{1} \in X$; this means that it is possible to add more of every good to any bundle in X and still remain in X.[1]

Given a preference relation $\succeq$ on X, let $d : \mathbf{R}^n_{++} \times \mathbf{R}_+ \to 2^X$ be the *demand correspondence* associated to $\succeq$; it is defined as

$$d(p,m) = \{x \in X : p \cdot x \leq m \text{ and } (\forall y \in X)(y \succ x \Longrightarrow p \cdot y > m)\}.$$

[1] This assumption is used to guarantee the existence of locally nonsatiated rationalizations.

We refer to d as a *demand function* if $d(p,m)$ is always a singleton.

A *consumption dataset* D is a collection (x^k,p^k), $k = 1,\dots K$, with $K \geq 1$ an integer, $x^k \in X$ and $p^k \in \mathbf{R}^n_{++}$. For each k, x^k is the consumption bundle purchased by the consumer at prices p^k. We shall assume that the consumer exhausts all his income, so that the expenditure $p^k \cdot x^k$ is also the total income devoted to consumption at the time at which the purchases were made. This assumption is in principle unavoidable, because if we were to allow for unspent (and unobservable) income, then any dataset is rationalizable. Section 3.2.3 discusses some related issues. As we shall see, the assumption that income equals expenditure amounts to an implicit assumption of local nonsatiation.

We seek to understand when the consumer's behavior is compatible with the basic theory of rational choice. Formally, a preference relation $\succeq$ *weakly rationalizes* a consumption dataset D if, for all k and $y \in X$, $p^k \cdot x^k \geq p^k \cdot y$ implies that $x^k \succeq y$. In other words, if d is the demand correspondence associated to $\succeq$, then $\succeq$ weakly rationalizes D if $x^k \in d(p^k, p^k \cdot x^k)$ for all k. Say that D is *weakly rationalizable* if there is a preference relation that weakly rationalizes D. There is a stronger version of rationalizability, which we shall discuss in Section 3.2.

We often specify a preference relation through a utility function. Say that a utility $u : X \to \mathbf{R}$ weakly rationalizes the data if for all k and $y \in X$, $p^k \cdot x^k \geq p^k \cdot y$ implies that $u(x^k) \geq u(y)$.

If we place no restrictions on $\succeq$, then any dataset is weakly rationalizable. We can just let $\succeq$ indicate indifference among all the elements of X. The resulting theory is not very interesting or useful: it explains everything because it predicts nothing. For this reason, we are going to impose some basic discipline on our exercise by requiring that the rationalizing $\succeq$ be monotonic.

Consider the situation in Figure 3.1. The figure illustrates a dataset $D_0 = \{(x^1,p^1),(x^2,p^2)\}$. The bundle x^2 is affordable at a budget in which x^1 was demanded. If the consumer were rational we could infer that she likes x^1 at least as much as x^2: we say that x^1 is *revealed preferred* to x^2. At the same time, x^2 is revealed preferred to x^1. We would conclude that a rational consumer who made these choices would regard x^1 and x^2 as equally good. Any rationalizing preference must impose indifference between x^1 and x^2.

Now we see that if the rationalizing preference is required to be monotone, then indifference is impossible. When the consumer in Figure 3.1 purchased x^1, she could have chosen x^2 by spending strictly less than she did at x^1. We should then conclude that she cannot regard the two choices as exactly equivalent: the consumer could have afforded a bundle $x^* \gg x^2$ at prices p^1 and expenditure $p^1 \cdot x^1$. By monotonicity of the rationalizing preference, the bundle x^* must be strictly preferred to x^2. At the same time, x^1 must be at least as good as x^*. So x^* must be strictly preferred to x^2; the consumer cannot be indifferent between the two. We infer that the observations in Figure 3.1 constitute a refutation of the hypothesis that the consumer is rational and his preferences are monotonic.

Generally, we say that a consumption dataset D satisfies the *weak axiom of revealed preference (WARP)* if there is no pair of observations (x^k,p^k) and (x^l,p^l) such that $p^k \cdot x^k \geq p^k \cdot x^l$ while $p^l \cdot x^l > p^l \cdot x^k$. In words, there is no pair

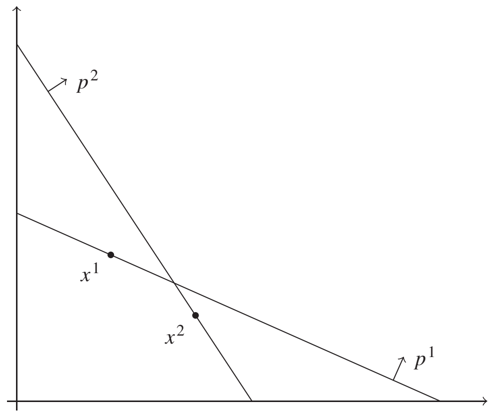


Fig. 3.1 A violation of WARP.

of observations such that the first is revealed preferred to the second, while the second is *strictly* revealed preferred to the first. The dataset in Figure 3.1 exhibits a violation of WARP (recall the discussion of WARP in 2.1.1).

To go beyond the example in Figure 3.1 it is useful to introduce some general definitions.

Given a consumption dataset $D = \{(x^k, p^k)\}_{k=1}^K$, we can define an order pair $\langle \succeq^R, \succ^R \rangle$ on X by $x \succeq^R y$ iff there is k such that $x = x^k$ and $p^k \cdot x^k \geq p^k \cdot y$; and $x \succ^R y$ iff there is k such that $x = x^k$ and $p^k \cdot x^k > p^k \cdot y$. Note that $\succ^R \subseteq \succeq^R$, but that $\succ^R$ will typically not be the strict binary relation associated to $\succeq^R$.[2] The pair $\langle \succeq^R, \succ^R \rangle$ is D's *revealed preference pair*. A dataset satisfies WARP iff there is no $x, y \in X$ with $x \succeq^R y$ and $y \succ^R x$.

Revealed preference pairs, as we have just defined them, are special cases of the notions introduced in Section 2.3 of Chapter 2. The primitives in Chapter 2 are: a set X, a collection of subsets $\Sigma \subseteq 2^X \backslash \{\varnothing\}$, and a choice function $c : \Sigma \to 2^X \backslash \{\varnothing\}$ such that for all $B \in \Sigma$, $c(B) \subseteq B$. Here, $X \subseteq \mathbf{R}^n_+$ is our given consumption space; the collection Σ of budget sets is defined as $\Sigma = \{B_k\}$, where $B_k = \{y \in X : p^k \cdot x^k \geq p^k \cdot y\}$; and $c(B) = \bigcup \{x^k : B_k = B\}$. Then the notion of revealed preference order pair coincides with the definition in Chapter 2, and the definition of WARP we have just given is equivalent to that of Chapter 2.

Observe that the relations $\succeq^R$ and $\succ^R$ are typically very incomplete, in the sense that many alternatives in X are not comparable according to these relations; the problem of rationalizability can be viewed as the problem of completing (or extending) these relations in a way that preserves transitivity.

We saw in the discussion of Figure 3.1 that WARP is necessary for weak rationalization by a monotonic preference. It will be shown below that WARP is not sufficient. The (stronger) property that is both necessary and sufficient for weak rationalization by a monotonic preference is defined as follows:

[2] This property of $\langle \succeq^R, \succ^R \rangle$ is in fact why we introduced the notion of order pairs.

A dataset satisfies the *generalized axiom of revealed preference* (GARP) if its revealed preference pair is an acyclic order pair. Note that the requirement that $\langle \succeq^R, \succ^R \rangle$ is an order pair implies that $\succ^R$ must be asymmetric. We also say that the revealed preference order pair satisfies GARP. Again, the concept of GARP corresponds to the definition in 2.3.

Define the *indirect revealed preference* relation to be the binary relation $(\succeq^R)^T$, the transitive closure of $\succeq^R$. In that sense, we can think of $\succeq^R$ as a *direct revealed preference* relation, and of $\succ^R$ as a *direct strict revealed preference* relation. Thus, one equivalent way to phrase GARP is to say that if x is indirectly revealed preferred to y, then y cannot be revealed strictly preferred to x.

GARP is also equivalent to the following property. For any sequence of data points $((x^{k_1}, p^{k_1}), \ldots, (x^{k_L}, p^{k_L}))$, if the following inequalities hold:

$$\begin{aligned}
p^{k_1} \cdot x^{k_1} &\geq p^{k_1} \cdot x^{k_2} \\
p^{k_2} \cdot x^{k_2} &\geq p^{k_2} \cdot x^{k_3} \\
&\vdots \\
p^{k_{L-1}} \cdot x^{k_{L-1}} &\geq p^{k_{L-1}} \cdot x^{k_L} \\
p^{k_L} \cdot x^{k_L} &\geq p^{k_L} \cdot x^{k_1},
\end{aligned}$$

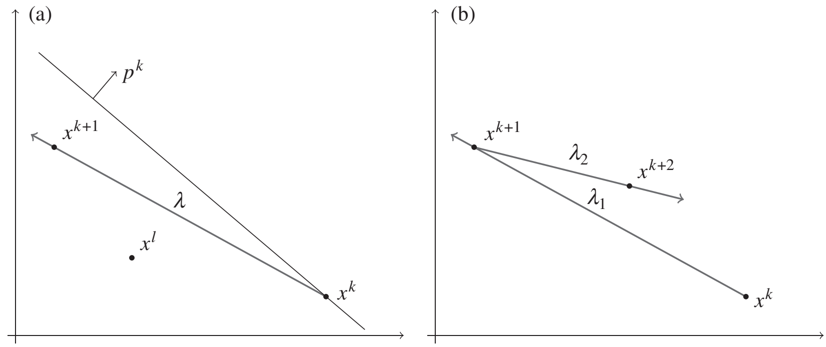
then they must hold with equality.

Theorem 3.1 *A dataset is weakly rationalizable by a monotonic preference relation iff its revealed preference pair satisfies GARP.*

With the identification made above between the primitives of this chapter and those of Chapter 2, Theorem 3.1 is easily seen to be a special case of Theorem 2.19. Simply verify that the conditions of Theorem 2.19 are satisfied; namely, 1) that each B_k is comprehensive with respect to the natural ordering $\langle \geq, > \rangle$; and 2) that $x \succ^R y$ iff $x \succeq^R y$ and there exists k with $x, y, z \in B_k$ $x \in c(B_k)$ and $z > y$. In fact, because all prices are strictly positive, the same properties hold with respect to the order pair $\langle \geq, \gg \rangle$.

Remark 3.2 The proof of Theorem 3.1 reveals a bit more than stated: GARP implies the existence of a strictly monotone rationalizing preference. Thus monotonicity and strict monotonicity are observationally equivalent. Further, we can impose local nonsatiation on our rationalizing preference. In that case, the existence of a locally nonsatiated weak rationalization implies GARP. So local nonsatiation and strict monotonicity are observationally equivalent.

Remark 3.3 When the consumption space is discrete, say it is $\mathbf{Z}^n_+$, then Theorem 3.1 must be modified. In this environment it is possible that $p^k \cdot x^l < p^k \cdot x^k$ but there is no $x \in \mathbf{Z}^N_+$ for which $x^l < x$ and $p^k \cdot x \leq p^k \cdot x^k$. For example, if $n = 2$, this occurs with the two-observation consumption dataset specified by $(x^1, p^1) = ((1,2),(4,3))$ and $(x^2, p^2) = ((2,0),(5,2))$. This consumption dataset



x^l lies below λ, then $x^k >^{\mathrm{R}} x^l$. $\quad$ x^{k+2} to the right of x^{k+1}.

Fig. 3.2 An illustration of the proof of Theorem 3.4.

is weakly rationalizable by a strictly monotonic preference, yet $p^1 \cdot x^2 < p^1 \cdot x^1$ and $p^2 \cdot x^1 < p^2 \cdot x^2$. The issue is that the notion $x^k \succ^R x^l$ as defined in Chapter 2 does not coincide with $p^k \cdot x^l < p^k \cdot x^k$ in this case.

The importance of the difference between WARP and GARP is brought out by our next results, Theorem 3.4 and Remark 3.5.

Theorem 3.4 *Suppose that there are exactly two goods ($n = 2$). Then a dataset satisfies GARP if and only if it satisfies WARP.*

Proof. We show that if a consumption dataset D fails GARP, then there must exist a cycle of length two. To this end, we will argue by contradiction and consider a cycle of minimal length, supposing that it has length at least three. The proof is geometric.

Suppose then that $(x^1,p^1),\ldots,(x^L,p^L)$ is a sequence of observations in D that imply a violation of GARP. So they form a cycle C:

$$x^1 \succeq^R x^2 \succeq^R \cdots \succeq^R x^L \succ^R x^1,$$

with $L \geq 3$. Suppose, toward a contradiction, that there are no shorter cycles.

We first show that no observed bundle is larger than another. Suppose to the contrary that $x^k \geq x^l$ for $k \neq l$. Working with indices modulo L, $x^{k-1} \succeq^R x^k$ would then imply $x^{k-1} \succeq^R x^l$, for $p^{k-1} \cdot x^{k-1} \geq p^{k-1} \cdot x^k \geq p^{k-1} \cdot x^l$. Then we may remove x^k, …, x^{l-1} from the cycle to form one of shorter length, retaining the strict inequality if it lies in this part of the cycle – a contradiction, as C was minimal.

So suppose that no observed bundle is larger than another. Since $n = 2$, the observed bundles can be ordered by the quantity of good 1 consumed in each bundle.

The basic observation is the following. Let x^k be a bundle in C. Consider the half-line in $\mathbf{R}^2_+$ starting from x^k and passing through x^{k+1}. See Figure 3.2(a). The half-line, named λ in the figure, will lie wholly within the budget at which

x^k was purchased. So if x^l is on or below the half-line, then $x^k \succeq^R x^l$. And $x^k \succ^R x^l$ when $x^k \succ^R x^{k+1}$, or when x^l is strictly below λ.

Now suppose that x^k has more of good 1 than x^{k+1}, so x^{k+1} lies to the left of x^k. Consider x^{k+2}, a bundle distinct from x^k and x^{k+1}. Let λ_1 be the half-line starting from x^k and passing through x^{k+1}, and λ_2 be the half-line starting from x^{k+1} and passing through x^{k+2}. Note that x^{k+2} cannot lie strictly below λ_1, or we would obtain $x^k \succ^R x^{k+2}$ and a shorter cycle than C.

We shall prove that x^{k+2} must be to the left of x^{k+1}. Suppose, toward a contradiction, that x^{k+2} has more of good 1 than x^{k+1}; so it lies to the right, as in Figure 3.2(b). If x^k were strictly below λ_2, we would have the cycle of length two $x^k \succeq^R x^{k+1} \succ^R x^k$. So the only possibility is that x^k lies on λ_2. Thus x^k, x^{k+1}, and x^{k+2} are all on the same line. But this configuration always leads to a shorter cycle than C, because it implies that $x^k \succeq^R x^{k+1} \succeq^R x^k$ and that $x^k \succeq^R x^{k+2}$. Thus, if $x^k \succ^R x^{k+1}$, or if $x^{k+1} \succ^R x^{k+2}$ (which implies $x^{k+1} \succ^R x^k$), then there is a shorter cycle – in fact a cycle of length two. And if $\neg(x^k \succ^R x^{k+1})$ and $\neg(x^{k+1} \succ^R x^{k+2})$ then the strict comparison in C lies elsewhere, so $x^k \succeq^R x^{k+2}$ leads to a shorter cycle. Thus x^{k+2} must be to the left of x^{k+1}.

The argument we have just made says that whenever x^{k+1} lies to the left of x^k, x^{k+2} must lie to the left of x^{k+1}. Now, C is a cycle, so (1) there is some k such that x^{k+1} is to the left of x^k; and (2) it cannot be true that x^{k+1} is to the left of x^k for all k. We are left with a contradiction.

Remark 3.5 Theorem 3.4 fails dramatically when $n \geq 3$. For each k there is an example of a dataset that is not weakly rationalizable, but that has no cycles of length less than k. Hence, when $n \geq 3$ there is no hope of simplifying GARP to some axiom that would only require ruling out cycles of certain lengths.

Consider the set $\{x \in \mathbf{R}^3_+ : \sum_i x_i = 1\}$. Draw on the face of this set a regular, convex polygon with k vertices (a regular k-gon). The vertices of the polygon will be the demand observations (these are vectors in $\mathbf{R}^3_+$). One needs to construct prices which result in a cycle of length k, but no shorter cycle. Consider going clockwise around the polygon. Any vertex, say observation x^j, has an adjacent vertex x^{j+1} (connected by an edge) in the clockwise direction. It is clear that one can separate the convex hull $\{x^j, x^{j+1}\}$ from the convex hull of $\{x^i\}_{i=1}^k \backslash \{x^j, x^{j+1}\}$. In fact, by a simple continuity argument, we can pick a hyperplane with normal vector q^j such that x^j lies on the hyperplane, $q^j \cdot x^{j+1} < q^j \cdot x^j$, and for all $i \notin \{j, j+1\}$, $q^j \cdot x^i > q^j \cdot x^j$. Then for α small enough, $p^j = \alpha q^j + (1, \ldots, 1)$ will be a vector of strictly positive prices for which $p^j \cdot x^{j+1} < p^j \cdot x^j$, and for all $i \notin \{j, j+1\}$, $p^j \cdot x^i > p^j \cdot x^j$ (recall that $\sum_{l=1,2,3} x^i_l = 1$ for all i). This gives a cycle of length k, but of no shorter length.

Theorem 3.1 answers the question of which datasets are weakly rationalizable. As we observed in Remark 3.2, consumption datasets cannot distinguish between a locally nonsatiated and a strictly monotonic preference relation. It turns out that much more can be said about the possible rationalizing preference relation: the following remarkable result says that the hypotheses of rationalization via a concave, strictly increasing, and continuous utility function are empirically indistinguishable from the hypothesis of rationalization via a locally non-satiated preference.

Afriat's Theorem *Let X be a convex consumption space, and $D = \{(x^k, p^k)\}_{k=1}^K$ be a consumption dataset. The following statements are equivalent:*

I) *D has a locally non-satiated weak rationalization;*
II) *D satisfies GARP;*
III) *There are strictly positive real numbers U^k and λ^k, for each k, such that*
$$U^k \leq U^l + \lambda^l p^l \cdot (x^k - x^l)$$
for each pair of observations (x^k, p^k) and (x^l, p^l) in D;
IV) *D has a continuous, concave, and strictly monotonic rationalization $u : X \to \mathbf{R}$.*

The inequalities in Statement III of Afriat's Theorem are called *Afriat inequalities*.

Remark 3.6 We can replace the system of inequalities in (III) by a smaller system, which only requires the inequalities to hold for k and l for which $p^l \cdot (x^k - x^l) \leq 0$. This fact is evident from the proof of the theorem in 3.1.1.

Before we give the proof, it is useful to interpret (III), the Afriat inequalities. The inequalities come from the first-order conditions for the maximization of a utility function subject to a budget set: Suppose for simplicity that the data is rationalizable by a utility function u that is differentiable. The first order condition would demand that

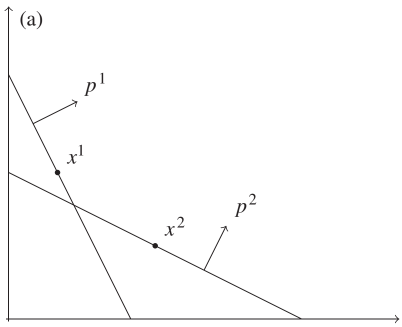
$$\nabla u(x)|_{x=x^k} = \lambda^k p^k,$$

where λ^k is a Lagrange multiplier. If, in addition, the utility function is concave we know that $u(y) - u(x) \leq \nabla u(x) \cdot (y - x)$ (see Proposition 1.11). By letting $U^k = u(x^k)$, one obtains the Afriat inequality

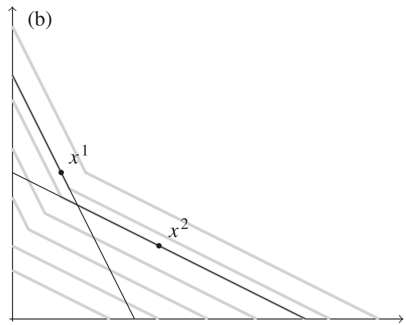
$$U^k - U^l = u(x^k) - u(x^l) \leq \nabla u(x^l) \cdot (x^k - x^l) = \lambda^l p^l \cdot (x^k - x^l).$$

It should then be clear that Afriat inequalities have two sources. One is satisfaction of the first-order condition for the consumer's problem. The other is the concavity of the utility function. The numbers U^k in (III) are meant to be the utility levels achieved when consuming x^k, and λ^k is meant to be the Lagrange multiplier.

What is remarkable about Afriat's Theorem is that GARP is sufficient to ensure a solution to this system of inequalities. The idea that GARP captures rationalizability (Theorems 2.19 and 3.1) is relatively easy to see. The result that concavity comes for free is deeper.



A dataset satisfying WARP.



Indifference curves for Afriat's utility.

Fig. 3.3 An illustration of the utility function in Afriat's Theorem.

Given a solution to the system of inequalities (III), one can write down a rationalizing utility function that satisfies the conditions in (IV):

$$u(x) = \min\{U^k + \lambda^k p^k \cdot (x - x^k) : k = 1, \dots, K\}.$$

This utility function is illustrated in Figure 3.3. On the left, in Figure 3.3(a) is a dataset satisfying WARP. On the right are indifference curves corresponding to the utility function defined, as above, from solutions to the Afriat inequalities for that dataset.

One last comment is in order. The utility we have just exhibited is not smooth (for a result on smooth utility, see Section 3.2.1); but we made use of smoothness when we introduced the first-order conditions. Smoothness is not crucial to the argument. The important concept is that $\lambda^k p^k$ must be a *supergradient* of the utility function. It is instructive to see why this is the case.

Assume given a solution to the Afriat inequalities. We can then string together Afriat inequalities as follows: $U^k - U^l \leq \lambda^l p^l \cdot (x^k - x^l)$ and $U^m - U^k \leq \lambda^k p^k \cdot (x^m - x^k)$ imply that

$$U^m - U^l \leq \lambda^l p^l \cdot (x^k - x^l) + \lambda^k p^k \cdot (x^m - x^k).$$

In fact, if we consider a sequence $k_1, k_2, \dots k_L$, and $k_L = k_1$, then

$$0 = \sum_{i=1}^{L-1} (U^{k_{i+1}} - U^{k_i}) \leq \sum_{i=1}^{L-1} \lambda^{k_i} p^{k_i} \cdot (x^{k_{i+1}} - x^{k_i}).$$

The property we have just derived from a solution to the Afriat inequalities is, in the terminology of 1.4, that the correspondence $\rho(x) = \bigcup_{k:x=x^k} \{\lambda^k p^k\}$ is *cyclically monotone* (on its domain). Then, reasoning as in Theorem 1.9 (and Corollary 1.10), one obtains a rationalizing utility.

In other words, one way of understanding Afriat's Theorem is as providing numbers λ^k for which the correspondence $\rho(x) = \bigcup_{k:x=x^k} \{\lambda^k p^k\}$ is cyclically monotone. In that sense, GARP can be viewed as an ordinal version of cyclic monotonicity.

Remark 3.7 One may similarly ask whether the existence of $\lambda^k > 0$ for which $\rho(x) = \bigcup_{k:x=x^k}\{\lambda^k p^k\}$ satisfies monotonicity is equivalent to WARP. Unfortunately it is not. An example is as follows. Let $p^1 = (1,0,0)$, $x^1 = (3,3,3)$, $p^2 = (0,1,0)$, $x^2 = (2,2,5)$, $p^3 = (0,0,1)$, and $x^3 = (4,0,4)$. It is easily verified that this consumption dataset satisfies WARP, and that there are no corresponding λ^k leading to a satisfaction of monotonicity. Prices can also be chosen strictly positive.

We offer two different proofs of Afriat's Theorem.

3.1.1 Proof of Afriat's Theorem

We prove that (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III) $\Longrightarrow$ (IV) $\Longrightarrow$ (I). That (I) $\Longrightarrow$ (II) follows from Theorem 3.1. That (IV) $\Longrightarrow$ (I) is obvious. We shall prove the other implications, starting from (II) $\Longrightarrow$ (III), which is the substance of this proof. The proof uses the Theorem of the Alternative, in the form of Lemma 1.12.

We shall prove that if there is no solution to the system of linear inequalities in (III), then GARP is violated. Using Lemma 1.12 it is easy to formulate the consequences of the inequalities not having a solution. We ignore the requirement that each $U^i > 0$, as any solution to the remaining inequalities can be made to satisfy this simply by adding a large enough constant to each U^i. We need to specify the matrix B that corresponds to these linear inequalities in a helpful way.

Consider all pairs $(i,j) \in \{1,...,K\}^2$ with $i \neq j$, and call this set of pairs A. We now construct a real-valued matrix of dimension $(|A|+K)\times 2K$. The first $|A|$ rows are indexed by elements of A, and the last K by $\{1,...,K\}$. The first K columns are labeled $1,...,K$, while the second K are labeled $1',...,K'$. The construction of the matrix is illustrated below.

$$
\begin{array}{c}
\\ (1,2) \\ \vdots \\ (i,j) \\ \vdots \\ \vdots \\ i \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|ccccc} 1 & \cdots & i & \cdots & j & \cdots & K & 1' & \cdots & i' & \cdots & K' \end{array} \\
\left[\begin{array}{ccccccc|ccccc}
1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & p^1\cdot(x^2-x^1) & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 1 & \cdots & -1 & \cdots & 0 & 0 & \cdots & p^i\cdot(x^j-x^i) & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \hline
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots
\end{array}\right]
\end{array}
$$

Define the matrix B as follows: For the first $|A|$ rows, in the row corresponding to (i,j), we put zeroes in all entries except for 1 in column i, -1 in column j, and $p^i\cdot(x^j-x^i)$ in column i'. We can write this succinctly by denoting a vector of all zeroes with a single 1 in its ith entry by $\mathbf{1}_i$. Then row (i,j) is the vector $\mathbf{1}_i - \mathbf{1}_j + p^i\cdot(x^j-x^i)\mathbf{1}_{i'}$. For the last K rows, in the row corresponding to i, we put the vector $\mathbf{1}_{i'}$. Clearly, a vector

$$(U,\lambda) = (U_1,\dots,U_K,\lambda_1,\dots,\lambda_K)$$

satisfies the system of inequalities in (III) iff $B_{(i,j)} \cdot (U,\lambda) \geq 0$, for all $(i,j) \in A$, and $B_i \cdot (U,\lambda) > 0$ for all i.

By Lemma 1.12, the nonexistence of a solution to Afriat's inequalities means that there is for each (i,j) a $\eta_{(i,j)} \geq 0$ and for each $h \in \{1,\dots,K\}$ a $\eta_h \geq 0$ such that

$$\sum_{(i,j)\in A} \eta_{(i,j)} B_{(i,j)} + \sum_{h=1}^{K} \eta_h B_h = 0, \tag{3.1}$$

and

$$\eta_h > 0 \qquad \text{for at least one } 1 \leq h \leq K. \tag{3.2}$$

Out of all $\eta = (\eta_{(i,j)}, \eta_h)$ satisfying (3.1) and (3.2), choose η to have a minimum number of entries that equal zero. Think of the numbers $\eta_{(i,j)}$ and η_h as weights. So Lemma 1.12 gives a weighted sum of rows that is identically zero, and where the weight is strictly positive for at least one of the last K rows of the matrix.

Let i be such that $\eta_i > 0$ as in (3.2). The weighted sum of entries in column i' is zero, so there must exist a pair (i,j) for which $p^i \cdot (x^j - x^i) < 0$ and $\eta_{(i,j)} > 0$. Then we have a positive weight on -1 in the column for j. So there must exist some k with $\eta_{(j,k)} > 0$. Then, writing the sum of entries in column j',

$$0 = \eta_j + \sum_{(j,k)} \eta_{(j,k)} p^j \cdot (x^k - x^j) = \eta_j + \sum_{\substack{(j,k)\\ \eta_{(j,k)}>0}} \eta_{(j,k)} p^j \cdot (x^k - x^j).$$

Since $\eta_j \geq 0$, it follows that there be some index k such that $p^j \cdot (x^k - x^j) \leq 0$ in addition to $\eta_{(j,k)} > 0$.

We therefore see that $x^i \succ^R x^j \succeq^R x^k$. In fact, what we proved above is the implication

$$\eta_{(i,j)} > 0 \Longrightarrow \text{there exists } k \text{ such that } \left\{\eta_{(j,k)} > 0 \text{ and } x^j \succeq^R x^k\right\}; \tag{3.3}$$

therefore, continuing in this fashion we obtain a cycle

$$x^{i_1} \succeq^R \cdots \succeq^R x^{i_L} \succeq^R x^{i_{L+1}} = x^{i_1}$$

as the set $\{x^k\}_1^K$ is finite.

There are two possibilities. Either $x^{i_a} \succ^R x^{i_{a+1}}$ for at least one a, or $p^{i_a} \cdot (x^{i_{a+1}} - x^{i_a}) = 0$ for all a. In the former case $\langle \succeq^R, \succ^R \rangle$ is not acyclic, so the data violate GARP and we are done.

In the latter case, we show that we can transform the weights η to obtain new weights η' such that $\eta'_{(i_a,i_{a+1})} = 0$ for at least one (i_a, i_{a+1}) with $p^{i_a} \cdot (x^{i_{a+1}} - x^{i_a}) = 0$. Note that, because $x^{i_1}, x^{i_2}, \dots, x^{i_L}, x^{i_1}$ is a cycle, and

$p^{i_a} \cdot (x^{i_{a+1}} - x^{i_a}) = 0$, we have that

$$\sum_{a=1}^{L} \left[(\mathbf{1}_{i_a} - \mathbf{1}_{i_{a+1}}) + p^{i_a} \cdot (x^{i_{a+1}} - x^{i_a}) \mathbf{1}_{i'_a} \right] = 0.$$

By construction (3.3) above, $\eta_{(i_a, i_{a+1})} > 0$ for each $1 \leq a \leq L$, so set

$$\kappa = \min_{a=1,\ldots,L} \eta_{(i_a, i_{a+1})}$$

and consider

$$\eta'_{(i,j)} = \begin{cases} \eta_{(i,j)} - \kappa & \text{if } \exists a : (i_a, i_{a+1}) = (i,j) \\ \eta_{(i,j)} & \text{otherwise,} \end{cases}$$

and $\eta'_i = \eta_i$ for all i. Then,

$$\begin{aligned} \sum_{(i,j)\in A} \eta'_{(i,j)} B_{(i,j)} + \sum_{i=1}^{K} \eta'_i B_i &= \sum_{(i,j)\in A} \eta_{(i,j)} B_{(i,j)} + \sum_{i=1}^{K} \eta_i B_i \\ &\quad - \kappa \left[\sum_{a=1}^{L} (\mathbf{1}_{i_a} - \mathbf{1}_{i_{a+1}}) + p^{i_a} \cdot (x^{i_{a+1}} - x^{i_a}) \mathbf{1}_{i'_a} \right] \\ &= 0. \end{aligned}$$

Note that η' equals zero for at least one (i_{a+1}, i_a) such that $p^{i_a} \cdot (x^{i_{a+1}} - x^{i_a}) = 0$ (the ones with minimum weight in η).

Therefore, $\eta' \geq 0$ is also a vector that satisfies Equations (3.1) and (3.2), $\eta' \leq \eta$, and η' has at least one entry $= 0$ that η does not have. This is not possible, as η was chosen to contain the minimal number of zeros. Thus, it must be the case $x^{i_a} \succ^R x^{i_{a+1}}$ for some $1 \leq a \leq L$, which proves (II) $\Longrightarrow$ (III).

To prove that (III) implies (IV), define a utility function $u : X \to \mathbf{R}$ by

$$u(x) = \min\{U^k + \lambda^k p^k \cdot (x - x^k) : k = 1, \ldots, K\}.$$

Observe that u is the minimum of continuous, monotone increasing, and concave (linear) functions, and hence is itself continuous, monotone increasing, and concave.

It is easy to see that u is a weak rationalization. First, $u(x^k) = U^k$ for all k, as (III) implies that $U^k = U^k + \lambda^k p^k \cdot (x^k - x^k) \leq U^l + \lambda^l p^l \cdot (x^k - x^l)$. Second, fix an observation k and let y be such that $p^k \cdot x^k \geq p^k \cdot y$. We have that $u(x^k) \geq u(y)$ because

$$u(x^k) = U^k \geq U^k + \lambda^k p^k \cdot (y - x^k) \geq u(y).$$

Thus the preference represented by u is a weak rationalization of the data. $\square$

3.1.2 Constructive proof of Afriat's Theorem

The substantive step in the proof of Afriat's Theorem is the implication (II) $\Longrightarrow$ (III). The proof we have given in 3.1.1 relies on Lemma 1.12, and has the advantage of illustrating a method that we use repeatedly in this book (see for instance Chapter 12; or Theorems 3.8 and 3.12 in the present chapter). We now turn to a proof that shows how one constructs numbers U^k and λ^k that solve the inequalities in (III). This proof has the advantage of giving a way of constructing a rationalizing utility by constructively finding a solution to the Afriat inequalities.[3]

Let $X_0 \subseteq X$ be the set of observed consumption bundles; that is, $X_0 = \{x^k : k = 1,\dots,K\}$. Consider the revealed preference pair $\langle \succeq^R, \succ^R \rangle$ restricted to X_0.

By Theorem 1.5, GARP implies that there is a preference relation $\succeq$ on X_0 such that $x \succeq y$ when $x \succeq^R y$ and $x \succ y$ when $x \succ^R y$. Partition X_0 according to the equivalence classes of $\succeq$. That is, let $I_1,\dots,I_J$ be a partition of X_0 such that $x \sim y$ for $x, y \in I_j$ and $x \succ y$ if $x \in I_j$, $y \in I_h$, and $j > h$.[4]

There are two important aspects of the partition $I_1,\dots,I_J$. First, if $x^k, x^l \in I_j$ then $p^l \cdot (x^k - x^l) \geq 0$ and $p^k \cdot (x^l - x^k) \geq 0$; as $p^k \cdot (x^l - x^k) < 0$, for example, would imply that $x^k \succ^R x^l$ and thus $x^k \succ x^l$. Second, if $x^k \in I_j$, $x^l \in I_h$ and $h < j$, then $p^l \cdot (x^k - x^l) > 0$; as $p^l \cdot (x^k - x^l) \leq 0$ would imply that $x^l \succeq^R x^k$ and thus $x^l \succeq x^k$.

We now define $(U^k, \lambda^k)_{k=1}^K$ recursively. Set $U^k = \lambda^k = 1$ if $x^k \in I_J$.

Suppose that we have defined (U^k, λ^k) for all $x^k \in \bigcup_{h=j+1}^J I_h$. We can choose V_j such that, for all $x^l \in I_j$ and $x^k \in \bigcup_{h=j+1}^J I_h$,

$$V_j < U^k \quad \text{and} \quad V_j < U^k + \lambda^k p^k \cdot (x^l - x^k). \tag{3.4}$$

Set $U^l = V_j$ for all l with $x^l \in I_j$.

Note that the choice of U^l ensures that if $x^k \in \bigcup_{h=j+1}^J I_h$ then $U^l < U^k$. Since $p^l \cdot (x^k - x^l) > 0$ for each $x^l \in I_j$, we can choose λ^l to be

$$\lambda^l = \max_k \frac{U^k - U^l}{p^l \cdot (x^k - x^l)} \geq 0 \tag{3.5}$$

where the max is taken over k such that $x^k \in \bigcup_{h=j+1}^J I_h$.

The chosen $(U^k, \lambda^k)_{k=1}^K$ satisfy the inequalities in (III). Indeed, let k and l be such that $x^k \in I_j$, $x^l \in I_h$ and $j > h$. Then (3.4) ensures that

$$U^l \leq U^k + \lambda^k p^k \cdot (x^l - x^k),$$

and (3.5) that

$$U^k \leq U^l + \lambda^l p^l \cdot (x^k - x^l). \tag{3.6}$$

If k and l are such that $x^k, x^l \in I_j$, then $U^k = U^l$ so (3.6) follows because $\lambda^k > 0$ and $p^l \cdot (x^k - x^l) \geq 0$.

[3] It should also be mentioned that linear programming methods, applied to the linear system in the proof in 3.1.1, can also be used to construct a solution to the Afriat inequalities.

[4] Strictly speaking, the existence of $\succeq$ is not constructive, as Theorem 1.5 relies on Zorn's Lemma. In the present case, however, X_0 is finite and $\succeq$ is easily constructed.

3.1.3 General budget sets

We turn to a version of Afriat's Theorem for general budget sets, budget sets that may not be defined by a vector of prices. We can generalize the statement $p^k \cdot (x - x^k) \leq 0$, specifying when x is affordable at prices p^k, by introducing a monotonic, continuous function $g^k : X \to \mathbf{R}$ for which $g^k(x^k) = 0$. We say that x is affordable when $g^k(x) \leq 0$.

A dataset is now a collection $(x^k, B^k), k = 1, \ldots, K$; where for each k there is a monotonic and continuous function $g^k : X \to \mathbf{R}$ such that $B^k = \{z \in X : g^k(z) \leq 0\}$. We suppose that $g^k(x^k) = 0$. Note that our previous notion of a dataset is a special case, where for each k, $g^k(x) = p^k \cdot (x - x^k)$.

Adapting the definitions of revealed preference, we can define $x \succeq^R y$ when there is k such that $x = x^k$ and $g^k(y) \leq 0$; and $x \succ^R y$ when there is k such that $x = x^k$ and $g^k(y) < 0$. A dataset satisfies GARP if $\langle \succeq^R, \succ^R \rangle$ is an acyclic order pair. A dataset satisfies WARP if $x \succeq^R y$ implies that $y \succ^R x$ does not hold.

In this more general environment, WARP and GARP are no longer equivalent when $n = 2$. Consider the three points $(0,3), (2,2), (3,0)$. For any pair of these, the convex and comprehensive hull of that pair does not include the third. We can choose our g^k functions so that there are three budget sets, the convex and comprehensive hulls of the pairs. It is then easy to construct a violation of GARP for which there is no corresponding violation of WARP.

The following generalization of Afriat's Theorem is due to Forges and Minelli.

Theorem 3.8 *Let X be convex. The following are equivalent:*

- I) *The data $\{(x^k, B^k)\}_{k=1}^K$ have a locally nonsatiated weak rationalization;*
- II) *$\langle \succeq^R, \succ^R \rangle$ is acyclic;*
- III) *There are strictly positive real numbers U^k and λ^k, for each k, such that*
$$U^l \leq U^k + \lambda^k g^k(x^l)$$
for each pair of observations k and l.
- IV) *The data $\{(B^k, x^k)\}_{k=1}^K$ have a monotonic and continuous rationalization;*

Further, in the case when all the functions g^k are concave, the statements above are equivalent to the existence of a monotonic, continuous, and concave rationalization.

Proof. First, that (IV) $\Longrightarrow$ (I) $\Longrightarrow$ (II) is immediate. The substance of the proof is (II) $\Longrightarrow$ (III), just as in the proof of Afriat's Theorem. The proof follows along the same lines as our proof of Afriat's Theorem.

To prove that (II) $\Longrightarrow$ (III), construct a matrix B as follows. Let A be the set of pairs (k,l) with $k \neq l$. For the first $|A|$ rows, in the row corresponding to (k,l), we put $\mathbf{1}_k - \mathbf{1}_l + g^k(x^l)\mathbf{1}_{k'}$. For the last K rows, in the row corresponding to k, we put the vector $\mathbf{1}_{k'}$. A vector

$$(U,\lambda) = (U^1,\ldots,U^K,\lambda^1,\ldots,\lambda^K)$$

satisfies the system of inequalities in (III) iff $B_{(k,l)} \cdot (U,\lambda) \geq 0$, for all $(k,l) \in A$, and $B_k \cdot (U,\lambda) > 0$ for all k.

Suppose that there is no solution to the system in (III). By Lemma 1.12 there is $\eta \geq 0$ such that

$$\sum_{(k,l)\in A} \eta_{(k,l)} B_{(k,l)} + \sum_k \eta_k B_k = 0,$$

and $\eta_k > 0$ for at least one k. Choose η to have the least number of zero entries.

Reasoning just as in the proof of Afriat's Theorem, we know that there is l for which $\eta_{(k,l)} > 0$ and $g^k(x^l) < 0$. The negative entry in the column for l in row (k,l) must cancel out with some row (w,l) with strictly positive weight in η; in fact w can be chosen such that $g^l(x^w) \leq 0$. Thus $x^k \succ^R x^l \succeq^R x^w$. The rest of the proof is analogous to the proof of Afriat's Theorem: we obtain a cycle of $\langle \succeq^R, \succ^R \rangle$.

To prove that (III) $\Longrightarrow$ (IV) we use a similar construction to the one in Afriat's Theorem; we let

$$u(x) = \min\{U^k + \lambda^k g^k(x) : k = 1,\ldots,K\}.$$

Note that u is monotone increasing and continuous. In addition, if $g^k(y) \leq 0$ then $u(y) \leq U^k = u(x^k)$; so it is a weak rationalization.

Finally, the construction above is concave when each g^k is concave.

3.2 STRONG RATIONALIZATION

A preference relation $\succeq$ *strongly rationalizes* a consumption dataset D if, for all k and $y \in X$,

$$(p^k \cdot y \leq p^k \cdot x^k \text{ and } y \neq x^k) \Longrightarrow x^k \succ y$$

Say that D is *strongly rationalizable* if there is a preference relation that strongly rationalizes D. Note that this definition implies choice must be single-valued, which is not the case for the notion of strong rationalization presented in 2.1. Otherwise, the definition of strong rationalization here is consistent with that in 2.1 under the assumption of single-valued choice.

Recall that one can understand weak rationalization as the extension of the observed demand behavior to a demand correspondence, namely the demand correspondence generated by the rationalizing preference. In contrast, strong rationalization is about extending the observed demand behavior to a (single-valued) demand function. In particular, the dataset $\{(x^1,p^1),(x^2,p^2)\}$, with $p^1 = p^2$, $x^1 \neq x^2$, and $p^1 \cdot x^1 = p^2 \cdot x^2$, is weakly rationalizable by a preference that is indifferent among x^1 and x^2, but it is not strongly rationalizable as it is incompatible with a single-valued demand function.

Given a consumption dataset D, we can define an order pair $\langle \succeq^S, \succ^S \rangle$ by $x \succeq^S y$ iff there is k such that $x = x^k$ and $p^k \cdot x^k \geq p^k \cdot y$; and $x \succ^S y$ iff $x \succeq^S y$ and $x \neq y$. The pair $\langle \succeq^S, \succ^S \rangle$ is D's *strong revealed preference* pair.

A dataset satisfies the *strong axiom of revealed preference* (SARP) if its strong revealed preference pair is acyclic.

Theorem 3.9 *A dataset is strongly rationalizable iff it satisfies SARP.*

Remark 3.10 The rationalization can, in fact, be assumed to be strictly monotonic, as in Theorem 3.1. Since we are considering strong rationalization, there is no need to impose monotonicity or local nonsatiation to rule out a trivial rationalization.

Proof. Let D be a dataset that satisfies SARP. We can define $\langle R^M, P^M \rangle$ from $\langle \succeq^S, \succ^S \rangle$ as $R^M = \succeq^S \cup \geq$ and $P^M = \succ^S \cup >$. Then the proof of the following lemma is similar to the proof of Theorem 2.19, and therefore omitted.

Lemma 3.11 *If $\langle \succeq^S, \succ^S \rangle$ satisfies SARP then $\langle R^M, P^M \rangle$ is acyclic.*

Now the proof follows from Theorem 1.5.

The next result is in the spirit of Afriat's Theorem. It shows that if we insist on a strong rationalization, then strict concavity is observationally equivalent to local nonsatiation. Note that strict concavity is trivially testable in the context of weak rationalization, as strict concavity (indeed strict quasiconcavity) of the utility function implies that demand is single valued. It turns out that strict concavity adds no additional empirical content to the hypothesis of single-valued demand.

Theorem 3.12 *Let X be a convex consumption space, and $D = (x^k, p^k)_{k=1}^K$ be a consumption dataset. Then the following are equivalent:*

I) *D has a locally nonsatiated strong rationalization.*
II) *D satisfies SARP.*
III) *There are strictly positive real numbers U^k and λ^k, for each k, such that for all k and l,*

$$U^k \leq U^l + \lambda^l p^l \cdot (x^k - x^l);$$

and further, if $x^k \neq x^l$,

$$U^k < U^l + \lambda^l p^l \cdot (x^k - x^l).$$

IV) *D has a continuous, strictly concave, and strictly monotonic strong rationalization $u : X \to \mathbf{R}$.*

Proof. The proof that (II) $\Longrightarrow$ (III) goes along familiar lines. Construct the matrix B as in Afriat's Theorem, but only include in A the pairs (i,j) with $p^i \cdot (x^j - x^i) \leq 0$ (as we remarked above, this can also be done in the proof of Afriat's Theorem). Now, partition the set A into A_1 containing the pairs

(k,l) with $x^k = x^l$, and A_2 containing the pairs (k,l) with $x^k \neq x^l$. Now, a vector $(U,\lambda) = (U^1,\dots,U^K,\lambda^1,\dots,\lambda^K)$ satisfies the inequalities in (III) iff $B_{(i,j)} \cdot (U,\lambda) \geq 0$, for all $(i,j) \in A_1$, $B_{(i,j)} \cdot (U,\lambda) > 0$, for all $(i,j) \in A_2$, and $B_i \cdot (U,\lambda) > 0$ for all i.

The nonexistence of a solution to this system of inequalities implies the existence of a vector of weights η such that

$$\sum_{(i,j)\in A_1} \eta_{(i,j)} B_{(i,j)} + \sum_{(i,j)\in A_2} \eta_{(i,j)} B_{(i,j)} + \sum_{i=1}^{K} \eta_i B_i = 0$$

with strictly positive weights $\eta_i > 0$ for some i, or $\eta_{i,j} > 0$ for some $(i,j) \in A_2$. If $\eta_i > 0$ then there is j such that $p^i \cdot (x^j - x^i) < 0$, and the proof proceeds to establish a cycle exactly as in the proof of Afriat's Theorem. If, instead, $\eta_{i,j} > 0$ for some $(i,j) \in A_2$ then $p^i \cdot (x^j - x^i) \leq 0$ and $x^i \neq x^j$ implies that $x^i \succ^R x^j$ by definition of $\langle \succeq^S, \succ^S \rangle$. Thus we start a cycle with a strict revealed preference comparison, just as in the case when $\eta_i > 0$.

The proof of (III) $\Longrightarrow$ (IV) requires more work than in the corresponding step of the proof of Afriat's Theorem. Let us fix a solution (U,λ) of the system (III).

Let $h : \mathbf{R}^n \to \mathbf{R}$ be a strictly convex and differentiable function such that $h(x) = 0$ only at 0, and is otherwise positive with globally bounded derivative (above and below) in all directions.[5] Let $\varepsilon > 0$ be small enough such that two criteria are satisfied: in the first place, such that

$$U^j < U^i + \lambda^i p^i \cdot (x^j - x^i) - \varepsilon h(x^j - x^i)$$

for all $x^i \neq x^j$; and in the second place, such that for all x and commodities i,

$$\varepsilon \nabla h(x) < \lambda^i p^i,$$

where $\leq$ is vector inequality and ∇ refers to the gradient (this is possible as h has bounded derivative in each direction and $\lambda^i p^i \in \mathbf{R}^n_{++}$).

We now define, for each i, $\varphi_i : X \to \mathbf{R}$ by

$$\varphi_i(x) = U^i + \lambda^i p^i \cdot (x - x^i) - \varepsilon h(x - x^i).$$

Finally, we define $u : X \to \mathbf{R}$ by

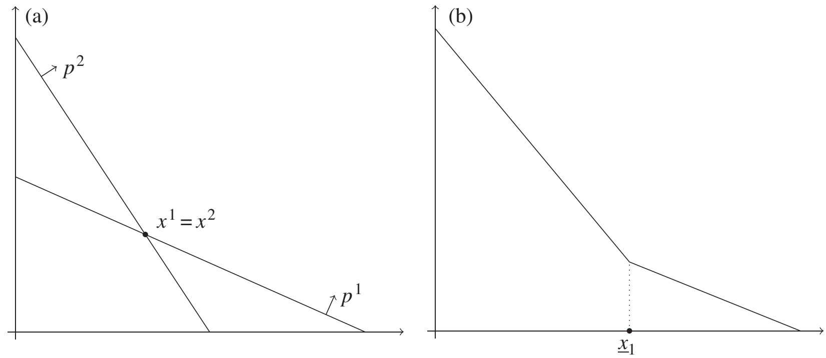
$$u(x) = \min_i \varphi_i(x).$$

It is obvious that u is strictly concave. We claim that it also rationalizes the data.

As a first step, we need to ensure that it is monotonic. This follows from the second property of ε above, which ensures that $\nabla \varphi_i \gg 0$.

Now, we must show that u rationalizes the data. First, note that $u(x^i) = \varphi_i(x^i) = U^i$ (this follows by the generalized Afriat inequalities in (III) and by

[5] For example, $h(x) = \sum_{i=1}^{n} (|x_i| - \ln(|x_i| + 1))$.



No smooth rationalization. Non-linear prices: a bulk discount.

Fig. 3.4 Smooth utility and general budget sets.

the choice of ε). Now, suppose that $p^i \cdot (x - x^i) \leq 0$ and $x \neq x^i$. Then

$$u(x) \leq \varphi_i(x) = U^i + \lambda^i p^i \cdot (x - x^i) - \varepsilon h(x - x^i) < U^i = u(x^i).$$

So u strongly rationalizes the data.

3.2.1 Smooth utility

Many economic models endow consumers with a differentiable utility, and it is natural to use consumption data to test for differentiability. Consider data $(x^1, p^1), (x^2, p^2)$ as in Figure 3.4(a); it is clear that a rationalizing utility would need to have a "kink" at x^1. The data in Figure 3.4(a) suggest a necessary condition for rationalization by a smooth function: that $x^k \neq x^l$ when $p^k \neq p^l$. This condition turns out to be the only added empirical content of smoothness, as formalized by the following result due to Chiappori and Rochet.

Theorem 3.13 *A dataset $D = (x^k, p^k)_{k=1}^K$ is strongly rationalizable by a smooth, strictly monotonic, and strictly concave utility $u : X \to \mathbf{R}$ iff D satisfies (1) SARP and (2) $x^k = x^l$ iff $p^k = p^l$.*

3.2.2 General budget sets

Finally, we discuss data with general budget sets. Let X be a convex set. A dataset is now a collection $(x^k, B^k), k = 1, \ldots, K$, where B^k is a closed set such that

I) the complement of B^k in X is a convex set;
II) if $z \notin B^k$ and $y > z$, then $y \notin z$;
III) if $y > x^k$, then $y \notin B^k$.

A consumer facing nonlinear prices is a classical example of how such data could be generated. For example, in the budget in Figure 3.4(b) there is a quantity discount: good 1 costs p_1 for quantities below $\underline{x}_1$ and $p_1' < p_1$ for quantities above.

Here (I) captures precisely the "quantity discount" idea, while (II) is a generalization of the monotonicity in standard budgets with linear prices. Requirement (III) corresponds to the standard assumption that choices exhaust all the consumer's income. A special case of this model is obtained when there are continuous, monotone increasing, and convex functions g^k such that $g^k(x^k) = 0$ and $B^k = \{y \in X : g^k(y) \leq 0\}$. It is interesting to contrast the model with the one in Section 3.1.3.

Adapting the definitions of revealed preference, we can define $x \succeq^S y$ when there is k such that $x = x^k$ and $y \in B^k$; and $x \succ^S y$ when $x \succeq^S y$ and $x \neq y$. By analogy with our previous theory, we say that the data satisfy SARP if $\langle \succeq^S, \succ^S \rangle$ is acyclic.

The following result is due to Rosa Matzkin.

Theorem 3.14 *Let X be convex and bounded. A dataset is strongly rationalizable by a strictly monotonic, and strictly concave utility iff it satisfies SARP.*

3.2.3 Partially observed prices and consumption

The analysis until now has assumed data on all relevant prices and consumptions. This assumption may not always be reasonable. For example the return on household labor, which can be seen as the price of leisure, may not be observable. Similarly, consumption surveys normally ask for information on consumption defined narrowly: supermarket purchases, purchases of non-durables, etc. The demand for all commodities may not be observable. We show in this section that such lack of observability leads to a serious lack of predictive power.

Let us begin with the case of partially observed prices. Say that we have $n+m$ commodities. We observe the consumption of all $n+m$ commodities, with K data points. However, we only observe the prices corresponding to the first n commodities.

Formally, we say that a *consumption dataset with partially observed prices* D is a collection (x^k, p^k), $k = 1, \ldots K$, with $x^k \in \mathbf{R}^{n+m}_+$ and $p^k \in \mathbf{R}^n_{++}$. We say that D, a consumption dataset with partially observed prices, is weakly rationalizable if for each $k = 1, \ldots, K$, there exists $q^k \in \mathbf{R}^m_{++}$ such that $(x^k, (p^k, q^k))$ is a consumption dataset which is weakly rationalizable.

Let x_n denote the projection of $x \in \mathbf{R}^{n+m}_+$ onto its first n coordinates; similarly define x_m to be the projection on the last m coordinates. Further, for a consumption dataset with partially observed prices, and $x \in \mathbf{R}^m_{++}$, we let $K_x = \{k \in \{1, \ldots, K\} : x^k_m = x\}$.

Theorem 3.15 *A consumption dataset with partially observed prices is weakly rationalizable by a locally nonsatiated preference iff for all $x \in \mathbf{R}^m_{++}$, the set $(x^k_n, p^k)_{k \in K_x}$ is weakly rationalizable by a locally nonsatiated preference.*

Proof. If the consumption dataset with partially observed prices is weakly rationalizable by a locally nonsatiated preference, there exist U^k, λ^k, and q^k such that for all $k,l \in K_x$, $U^k \leq U^l + \lambda^l(p^l,q^l) \cdot (x^k - x^l)$. But since $x^k_m = x = x^l_m$, these inequalities become $U^k \leq U^l + \lambda^l p^l \cdot (x^k_n - x^l_n)$, so that the Afriat inequalities are satisfied for $(x^k_n, p^k)_{k \in K_x}$.

Conversely, suppose that each $(x^k_n, p^k)_{k \in K_x}$ is weakly rationalizable by a locally nonsatiated preference. Let $q \in \mathbf{R}^m_{++}$ be such that $x^k_m \neq x^l_m$ implies $q \cdot x^k_m \neq q \cdot x^l_m$; simply choose q to be not perpendicular to any of the vectors $\{x^k_m - x^l_m\}_{k \neq l}$.

Now, for each k, let $\mu^k > 0$ be large enough so that

$$\mu^k > \max_{\{l : x^l_m \neq x^k_m\}} \left\{ \frac{p^k \cdot (x^l_n - x^k_n)}{q \cdot (x^k_m - x^l_m)}, 1 \right\}.$$

Let $q^k = \mu^k q$. Let us define our pair $\langle \succeq^R, \succ^R \rangle$ as usual, for the consumption dataset $(x^k, (p^k, q^k))$. There can be no cycles within K_x for any x, by hypothesis. We now claim that for k,l for which $x^k_m \neq x^l_m$, $q \cdot x^k_m > q \cdot x^l_m$ implies that $x^l \succeq^R x^k$ is false. To see this, $q^l \cdot (x^l_m - x^k_m) < p^l \cdot (x^k_n - x^l_n)$, so that $(p^l, q^l) \cdot x^l < (p^l, q^l) \cdot x^k$.

Now, we claim that there are no $\langle \succeq^R, \succ^R \rangle$ cycles. Suppose for a contradiction that there is a cycle, $x^1 \succeq^R x^2 \succeq^R \ldots \succeq^R x^K \succ^R x^1$. Then we claim that for all k, $q \cdot x^k_m \geq q \cdot x^{k+1}_m$, where addition is modulo K as usual. If not then there is a k for which the inequality is reversed. In this case, we must have $x^k_m \neq x^{k+1}_m$ and $q \cdot x^{k+1}_m > q \cdot x^k_m$, from which we obtain from the previous paragraph that $x^k \succeq^R x^{k+1}$ is false, which is a contradiction. Consequently, we have that for all k, $q \cdot x^k_m = q \cdot x^{k+1}_m$, from which we obtain by the assumption on q that $x^k_m = x^{k+1}_m$. This implies that all elements of the cycle must lie in a single K_x, which was assumed false. Hence there is no $\langle \succeq^R, \succ^R \rangle$ cycle.

Importantly, Theorem 3.15 demonstrates that if demand for commodities whose prices are unobserved is always distinct across observations, preference maximization implies no additional conditions.

We can also ask what happens when there is unobserved consumption. It turns out that as soon as there is some commodity whose consumption we cannot observe, there is no empirical content to preference maximization. A *consumption dataset with partially observed consumption D* is a collection (x^k, p^k) for $k = 1, \ldots, K$, where each $x^k \in \mathbf{R}^n_+$, and each $p^k \in \mathbf{R}^{n+m}_{++}$.

Theorem 3.16 *Suppose that $m \geq 1$. Then for any consumption dataset with partially observed consumption, there exists $y^k \in \mathbf{R}^m_+$ for which $((x^k, y^k), p^k)$, $k = 1, \ldots, K$, is a consumption dataset which is weakly rationalizable by a locally nonsatiated preference.*

Proof. The indices naturally order observations. We assume zero consumption for all commodities with index at least as high as $n+2$. Consumption of commodity $n+1$ is defined first for $k = 1$ as 0. Inductively it is defined to be large enough so that $\{x : p^k \cdot x \leq p^k \cdot (x^k_n, x^k_{n+1}, 0, \ldots, 0)\}$ is nested in the relative interior of $\{x : p^{k+1} \cdot x \leq p^{k+1} \cdot (x_n^{k+1}, x_{n+1}^{k+1}, 0, \ldots, 0)\}$. To do so, note that the set $\{x : p^k \cdot x \leq p^k \cdot (x_n^k, x_{n+1}^k, 0, \ldots, 0)\}$ is compact, so we can maximize $p^{k+1} \cdot x$ on it, and let the maximum value of this be w. Choose x_{n+1}^{k+1} so that $p^{k+1} \cdot (x_n^{k+1}, x_{n+1}^{k+1}, 0, \ldots, 0)$ is greater than w, and set $y^k = (x_{n+1}^{k+1}, 0, \ldots, 0)$. By this construction, we have that for $k > l$, $p^k \cdot (x^k, y^k) > p^k \cdot (x^l, y^l)$. Conversely, suppose that $p^k \cdot (x^k, y^k) > p^k \cdot (x^l, y^l)$. Clearly $l \neq k$. If $l > k$, then by construction, $p^k \cdot (x^l, y^l) > p^k \cdot (x^k, y^k)$; since y^l was chosen so that (x^l, y^l) is not in the set $\{x : p^k \cdot x \leq p^k \cdot (x^k, y^k)\}$. So $k > l$ iff $p^k \cdot (x^k, y^k) > p^k \cdot (x^l, y^l)$. Consequently, the data $((x^k, y^k), p^k)$ can be rationalized by a locally nonsatiated preference.

Theorems 3.15 and 3.16 allow flexibility in choosing the observed wealth, which is something which, in the setup of this chapter, has been taken to be equal to expenditure (hence determined, or observed). If wealth is observable, these results break down. It remains an open question to characterize the empirical content of preference maximization with only partially observed prices and/or consumption, especially when wealth can be observed. Results exist for the one-dimensional case where bounds are put on consumption in Theorem 3.16.

Finally, we note that empirical studies of consumption avoid the conclusion of Theorem 3.16 by assuming that preferences are separable. If one studies, for example, supermarket purchases, then one assumes that goods which are not purchased in a supermarket are separable from the ones that are. This means that if an agent conforms to the theory, she will have a budget for the supermarket. One can then treat the problem of choosing optimal consumption in the supermarket separately from the rest of the agent's consumption decisions. Separability is treated in 4.2.2.

3.3 REVEALED PREFERENCE GRAPHS

Finally, we investigate here a question which is related specifically to the structure of linear budget sets. Return to the notion of data introduced in 3.1: a collection (x^k, p^k), $k = 1, \ldots K$, with $K \geq 1$ an integer, $x^k \in X$ and $p^k \in \mathbf{R}^n_{++}$. The integer K is the cardinality of the dataset.

For a consumption dataset D of cardinality K, we can define the *revealed preference graph* $\langle \trianglerighteq, \triangleright \rangle$ on $\{1, \ldots, K\}$ by $i \trianglerighteq j$ if $p^i \cdot x^i \geq p^i \cdot x^j$ and $i \triangleright j$ if $p^i \cdot x^i > p^i \cdot x^j$. We say that $\langle \trianglerighteq, \triangleright \rangle$ is the revealed preference graph defined by D.

The following simple result states that there are no *a priori* restrictions on which order pairs can be revealed preference graphs.

Theorem 3.17 *Let $\langle \trianglerighteq, \triangleright \rangle$ be an order pair in which $\trianglerighteq$ is a reflexive binary relation and $\triangleright$ is an irreflexive binary relation on $\{1, \ldots, K\}$. Then there exists X and a consumption dataset D on X of cardinality K for which $\langle \trianglerighteq, \triangleright \rangle$ is the corresponding revealed preference graph.*

Proof. Suppose $\trianglerighteq$ is reflexive. Let $X = \mathbf{R}^K_+$ and let $x^i = \mathbf{1}_i$; the unit vector in dimension i. For each i and j, set

$$p^i_j = \begin{cases} 1/2 & \text{if } i \triangleright j \\ 1 & \text{if } i \trianglerighteq j \text{ and not } i \triangleright j \\ 2 & \text{if not } i \trianglerighteq j \end{cases}$$

It is easy to verify that D so defined has $\langle \trianglerighteq, \triangleright \rangle$ as its revealed preference order pair.

Theorem 3.17 allows flexibility in choosing consumption space. If consumption space is fixed, then the question of which order pairs can be revealed preference graphs remains open. We will encounter phenomena of this type several times; see for example Theorem 9.1 and Theorem 11.2.

Theorem 3.18 *For any $X \subseteq \mathbf{R}^n_+$ in which $n \geq 2$, for any $K \geq 1$, the following order pairs are revealed preference graphs on $\{1, \ldots, K\}$:*

I) $\langle = \cup \neq, \neq \rangle$
II) $\langle =, \varnothing \rangle$

In the first graph of Theorem 3.18, every observation is strictly revealed preferred to every other observation.

Proof. To see that the first is a revealed preference graph, let $\mathcal{S}$ be the unit sphere in $\mathbf{R}^n$. Fix arbitrary $p^1, \ldots, p^K \in \mathbf{R}^n_{++}$ such that p^i and p^j are not collinear whenever $i \neq j$, define $x^i = \arg\max_{\mathcal{S}} p^i \cdot x$. It is easy to verify that this data has $\langle = \cup \neq, \neq \rangle$ as its revealed preference graph. (Note that any smooth convex set with nonempty interior intersecting the strictly positive orthant will work in place of $\mathcal{S}$, so long as we can choose p^i so to have unique and distinct maximizers in the positive orthant.)

Showing that $\langle =, \varnothing \rangle$ is a revealed preference graph follows a similar construction, instead letting $\mathcal{P} = \{x \in \mathbf{R}^n_+ : \prod x_i \geq 1\}$. Choose arbitrary p^i (as long as p^i and p^j are not collinear when $i \neq j$) and let $x^i = \arg\min_{\mathcal{P}} p^i \cdot x$. Similarly, there is nothing particularly special about the set $\mathcal{P}$.

3.4 CHAPTER REFERENCES

Samuelson (1938) introduced the idea of revealed preferences, and formulated the weak axiom of revealed preference. Little (1949) and Samuelson (1948) showed how to construct indifference curves from revealed preference in the two-commodity case. Houthakker (1950) proved a version of Theorem 3.1, the first characterization of the empirical content of rational consumption. Houthakker's condition was stated for single-valued demand, and was equivalent to the condition we here call SARP. Related are the papers by

Newman (1960) and Stigum (1973).[6] Samuelson (1950) coined the phrase "strong axiom of revealed preference;" however, the concept he attributes to Houthakker is slightly different from Houthakker's actual contribution. Specifically, Samuelson required that the strong revealed preference pair be quasi-acyclic. There seems to be no general consensus on the meaning of SARP in the literature; for more on this, see Suzumura (1977). We should also mention the contribution of Mas-Colell (1978), which can be viewed as an identification result for demand functions satisfying certain revealed preference axioms.

Marcel Richter, in two important survey articles (Richter, 1971 and Richter, 1979) presents a unified view of revealed preference, discussing the commonalities and differences between the material in Chapters 2 and 3. In particular, Richter (1979) accounts for how the duality between choices and budgets translates into revealed preference theory. Richter also covers the relation to the question of integrability of demand, which is out of the scope of our book.

The observation in Remark 3.3 is due to Polisson and Quah (2013).

Theorem 3.4 is due to Rose (1958), though the result was conjectured earlier by Hicks (1956, pp. 52–54).[7] An example of a well-defined, continuous, almost everywhere differentiable demand function satisfying WARP and not GARP is given in Gale (1960a) (see also Hicks, 1956, pp. 110–111). Kihlstrom, Mas-Colell, and Sonnenschein (1976) provide a theory of such demand functions. WARP is the absence of cycles of length two: a natural conjecture is that absence of cycles of length n would suffice for rationalizability in n-dimensional space. Kamiya (1963) gives an example of finite data for three consumption goods which exhibits no cycles of length two or three, but which is not rationalizable. Shafer (1977) establishes that this non-equivalence is robust. Shafer works with demand functions (not finite data) and shows that for three commodities, one must require absence of cycles of all possible lengths, by exhibiting, for any n, single-valued demand functions which exhibit no cycles of length n, but which are not rationalizable. Peters and Wakker (1994, 1996) discuss this result for environments with more than three commodities; see also John (1997). Finally, a recent investigation into these ideas is Heufer (2007).

Afriat's Theorem appeared first in Afriat (1967), but was popularized by Hal Varian in a collection of papers ((1982; 1983a; 1985)). Recently, Fostel, Scarf, and Todd (2004) and Chung-Piaw and Vohra (2003) provided new proofs of Afriat's Theorem. Our proof in 3.1.1 is different from these, and follows basically the argument by Matzkin and Richter (1991). The first

[6] Uzawa (1960a, 1971) investigates demand functions satisfying WARP and a regularity condition. See also Bossert (1993). A series of papers investigates a related "continuous" version of an acyclicity condition; see Ville (1946); Ville and Newman (1951–1952); Hurwicz and Richter (1979).

[7] See also Newman (1955); Houthakker (1957); Newman (1960), Blackorby, Bossert, and Donaldson (1995), and Banerjee and Murphy (2006).

linear-programming-style proofs of Afriat's Theorem are due to Diewert (1973). The argument using linear combinations of rows in the proof is a form of the Poincaré–Veblen–Alexander Theorem (see Berge, 2001; the theorem is attributed to early contributors to the mathematical discipline of topology, i.e. Poincaré, 1895; Veblen and Alexander, 1912–1913). The constructive proof of Afriat's Theorem in 3.1.2 is taken from Varian (1982).

The recent paper by Reny (2014) considers infinite datasets, and shows that GARP is necessary and sufficient for the existence of a quasiconcave rationalization. Reny's result unifies the approaches based on finite datasets and demand functions. It is noteworthy that an infinite dataset satisfying GARP may not be rationalizable by a concave utility, only a quasiconcave one. The theorem of Reny also gives a particularly simple proof of a weak version of Afriat's Theorem (one ensuring a quasiconcave rationalization) for finite data. Sondermann (1982) provides a sufficient condition ensuring representation by an upper semicontinuous (closed upper contour sets) utility.

Necessary and sufficient conditions for the existence of λ^k such that $x^k \mapsto \lambda^k p^k$ is monotone are presented in John (2001), where it is also shown that the model is equivalent to rationalization by a nontransitive preference with a certain type of "utility" rationalization which has a certain convexity property.

Probably the first paper to study a version of Afriat's Theorem for nonlinear budget sets is Yatchew (1985), who investigates necessary and sufficient conditions for convex utility maximization on finite unions of polyhedra in the form of inequalities. Theorem 3.8 is from Forges and Minelli (2009). Forges and Minelli also present a stronger result, where if the functions g^k are quasiconvex and differentiable then there is a concave rationalization, if there is a rationalization. Cherchye, Demuynck, and De Rock (2014) build on this result, establishing a characterization of datasets which are weakly rationalizable by quasiconcave, locally nonsatiated, and continuous preferences on arbitrary budget sets which are closed and satisfy free disposability.

Theorem 3.12 is due to Matzkin and Richter (1991). Matzkin and Richter clarify the notions of weak and strong rationalizability, and put the previous work on revealed preference in this context. Theorem 3.13 is due to Chiappori and Rochet (1987). Matzkin and Richter (1991) also present a result on smooth rationalization, in the spirit of Theorem 3.13. The result on general budget sets in Theorem 3.14 is due to Matzkin (1991).

Theorems 3.15 and 3.16 are due to Varian (1988a); the case of $m = 1$ is established there.

Theorem 3.17 can be found in Deb and Pai (2014), as well as the first part of Theorem 3.18, though these results are implicit in other works.

Finally, we have not discussed stochastic choice in demand theory: See the papers by Bandyopadhyay, Dasgupta, and Pattanaik (1999) and Bandyopadhyay, Dasgupta, and Pattanaik (2004).