CHAPTER 4

Topics in Rational Demand

Chapter 3 established the testable implications of the hypothesis that consumers are rational. We are often interested in situations where the rationalizing preference, or demand function, satisfies additional properties. We want to know what additional structure is imposed on a dataset from demanding that the rationalization has to use a utility function with some given property. In particular, we focus on the properties of supermodularity, submodularity, homotheticity, separability, complements, and substitutes.

4.1 DISCRETE GOODS: SUPERMODULAR AND SUBMODULAR RATIONALIZATIONS

In Chapter 3 we discussed a collection of results in the spirit of Afriat's Theorem. In these results, one obtains a concave rationalization from the rationalizability of the data. Arguably, though, most consumption goods come in discrete units. Some goods seem particularly "lumpy," such as cars and houses. For such goods, the notion of concavity is not well defined. One can instead investigate super- and submodularity. Supermodularity corresponds to the notion that goods are complements: specifically, that increases in the consumption of one good become more valuable when one consumes more of the other goods. Submodularity corresponds to the property of substitute goods.

The meaning of super- and submodularity can be understood from Figure 4.1. In the figure, there are two goods. For any two bundles x and y, $x \wedge y$ is the component-wise minimum of the bundles x and y, and $x \vee y$ is the component-wise maximum (see the definitions in Chapter 1). The function $u : X \to \mathbf{R}$ is *supermodular* if for all $x, y \in X$,

$$u(x) + u(y) \leq u(x \vee y) + u(x \wedge y),$$

and *submodular* if $-u$ is supermodular.

If u is supermodular then the change in utility $u(x) - u(x \wedge y)$ cannot exceed the change $u(x \vee y) - u(y)$. Note that in Figure 4.1, the increase in good 2 when we go from $x \wedge y$ to x is the same as when we go from y to $x \vee y$. This means that

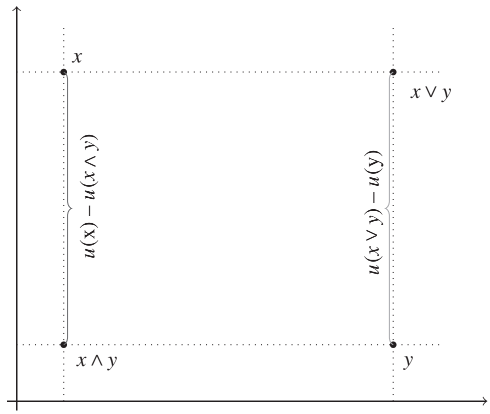


Fig. 4.1 Supermodularity.

the change in utility resulting from adding the amount of good 2 in the change from $x \wedge y$ to x, can only be larger when the quantity of good 1 is larger. This is a notion of complementary goods: the increases in utility due to increases in consumption of good 2 are larger as we consume more of good 1.

Submodularity has the opposite interpretation. If u is submodular in the figure, then the increase in utility due to the increase in consumption of good 2 is diminished by a higher consumption of good 1. Thus the two goods are substitutes.

It turns out that, when goods are discrete, supermodularity and submodularity of utility are empirically indistinguishable from rationalizability of the data. This result is in the spirit of Afriat's, but it comes from a rather different set of ideas.

Let $X \subseteq \mathbf{Z}^n_+$. A *(discrete) dataset* is a finite list of observations (x^k, B^k), $k = 1, \ldots, K$, where for each k, $x^k \in B^k \subseteq X$. The interpretation is that x^k is the chosen element from the budget B^k. Assume:

- x^k is maximal in B^k, in the sense that if $y > x^k$ then $y \notin B^k$.
- B^k satisfies the property that $z \in B^k$ whenever there is $y \in B^k$ and $z \leq y$.
- $|B^k| < +\infty$.

Given a dataset D, we can define the *strong revealed preference* pair $\langle \succeq^S, \succ^S \rangle$ as in Chapter 3 by $x \succeq^S y$ iff there is k such that $x = x^k$ and $y \in B^k$, and $x \succ^S y$ iff $x \succeq^S y$ and $x \neq y$. Then the *strong axiom of revealed preference* (SARP) is the requirement that $\langle \succeq^S, \succ^S \rangle$ is acyclic.

Note that Theorem 3.9 is valid for the model discussed here. The notion of data is different, but the assumptions we have made on the data guarantee that the proof of the theorem applies as written.

Theorem 4.1 *Let $X \subseteq \mathbf{Z}^n_+$ be a lattice. The following statements are equivalent:*

I) *D is strongly rationalizable.*
II) *D satisfies SARP.*
III) *D has a strictly monotonic and supermodular strong rationalization* $u : X \to \mathbf{R}$.
IV) *D has a strictly monotonic and submodular strong rationalization* $v : X \to \mathbf{R}$.

Proof. The equivalence of (I) and (II) is clear from previous results (Theorem 3.9). We need to prove that (II) implies (III) and (IV).

Because K is finite, and each B^k is finite, there is M such that for all $x \in \bigcup_k B^k$ and all i, $x_i < M$. Let $X_M = \{x \in \mathbf{Z}^n_+ : x_i < M \text{ for all } i\}$.

Let D satisfy SARP. Imagine first that consumption space is X_M. By Theorem 3.9, D has a strictly monotonic rationalization $\succeq$ on X_M. Since X_M is finite, there is an integer-valued utility function $v : X_M \to \mathbf{Z}_+$ defined by $v(x) = |\{y \in X_M : x \succeq y\}|$. Let $V = \sup_{x \in X_M} v(x)$. Define $g : X \to \mathbf{R}$ as follows. For $x \in X_M$, $g(x) = v(x)$. Otherwise, $g(x) = V + \sum_i x_i$. Note that g is a strictly monotonic, integer-valued function which strongly rationalizes the data.

Let $u(x) = 2^{g(x)}$; we claim that u is a supermodular strong rationalization of D. It is clearly a strong rationalization because it is a monotonic transformation of g. To see that it is supermodular, let x and y in X and suppose without loss of generality that $g(x) \geq g(y)$. If $x = x \vee y$ then $y = x \wedge y$ so there is nothing to prove. Suppose then that $x \neq x \vee y$, which implies that $x \vee y \succ x$, as $\succeq$ is strictly monotonic. Thus $g(x \vee y) \geq g(x) + 1$. Then,

$$\begin{aligned} u(x) + u(y) &= 2^{g(x)} + 2^{g(y)} \leq 2^{g(x)} + 2^{g(x)} \leq 2^{g(x)+1} \\ &\leq 2^{g(x \vee y)} = u(x \vee y) \leq u(x \vee y) + u(x \wedge y); \end{aligned}$$

so u is supermodular.

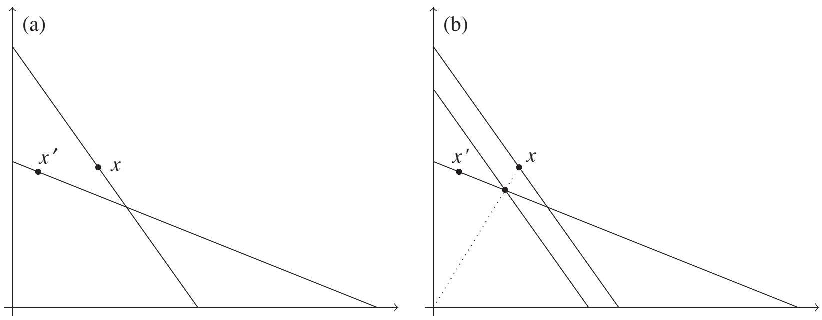
To exhibit a submodular strong rationalization, define $v : X \to \mathbf{R}$ by $v(x) = \sum_{i=0}^{g(x)} 2^{-i}$. The function v is a strong rationalization of D because v is a monotonic transformation of g. To see that v is submodular, suppose that $x \neq x \vee y$ and $x \neq x \wedge y$ (otherwise we have nothing to prove). Then,

$$v(x) - v(x \wedge y) = \sum_{j=g(x \wedge y)+1}^{g(x)} 2^{-j} \geq \left(\frac{1}{2}\right)^{g(x \wedge y)+1}.$$

On the other hand,

$$\begin{aligned} v(x \vee y) - v(y) &= \sum_{j=g(y)+1}^{g(x \vee y)} 2^{-j} \leq \sum_{j=g(y)+1}^{\infty} 2^{-j} \\ &= \left(\frac{1}{2}\right)^{g(y)} \leq \left(\frac{1}{2}\right)^{g(x \wedge y)+1} \end{aligned}$$

where the last inequality follows because $x \neq x \vee y$ implies that $y \neq x \wedge y$, so $y \succ x \wedge y$ by the strict monotonicity of $\succeq$. Thus $v(x \vee y) - v(y) \leq v(x) - v(x \wedge y)$, which shows that v is submodular.



Data $\{(p, x), (p', x')\}$ satisfying GARP.   Projected data violate GARP.

Fig. 4.2 Testable implications of homothetic preferences.

4.2 DIVISIBLE GOODS

Super- and submodularity are properties of particular interest when studying discrete goods. Other properties are commonly studied in environments with divisible goods. We now turn to homotheticity, separability, complements, and substitutes.

Assume throughout that consumption space $X \subseteq \mathbf{R}^n_+$ is a convex set. The notion of dataset is the same as in 3.1: A *consumption dataset D* is a collection (x^k, p^k), $k = 1, \dots K$, with $K \geq 1$ an integer, $x^k \in X$ and $p^k \in \mathbf{R}^n_{++}$.

4.2.1 Homotheticity

Here we assume X is a *cone*, so that whenever $x \in X$ and $\alpha \geq 0$, we have $\alpha x \in X$. A preference $\succeq$ is *homothetic* if, for all $\alpha > 0$ and all x, y, $x \succeq y$ implies that $\alpha x \succeq \alpha y$. A function $u : X \to \mathbf{R}$ is *homothetic* if the preference that it represents is homothetic.

We can begin to understand the testable implications of homotheticity from Figure 4.2. Suppose that we are interested in whether the data $\{(p,x), (p',x')\}$ have a homothetic rationalization. Clearly, the data satisfy GARP so they have a rationalization. It is easy to see, however, that no homothetic rationalization exists. The reason can be gleaned from Figure 4.2(b): a homothetic rationalization would imply that demand would have to lie on the ray joining x and 0, for any budget line that is parallel to the budget line for (p,x). So if we "deflate" the budget line for (p,x) until it crosses the point where the ray crosses the budget line for (p',x') the demand would have to lie at that intersection. We would then have a violation of WARP.

Somewhat more formally, Figure 4.2 presents the following situation. If there were a homothetic rationalization, then any supporting hyperplane of the upper contour set $\{y \in X : y \succeq x\}$ would also support $\{y \in X : y \succeq \theta x\}$, for $\theta > 0$. Choose θ such that θx lies on the budget line for (p',x'), i.e. $p' \cdot (\theta x) = p' \cdot x'$; thus x' is revealed preferred to θx. By homotheticity, p supports the upper contour set of $\succeq$ at θx, so θx should be chosen at prices p and income $p \cdot \theta x$. Then WARP would require that $p \cdot (\theta x) \leq p \cdot x'$.

The reasoning above leads to the following necessary condition for rationalization by a homothetic preference:

$$\theta = \frac{p' \cdot x'}{p' \cdot x} \Longrightarrow \theta p \cdot x \leq p \cdot x';$$

or,

$$1 \leq \frac{p \cdot x'}{p \cdot x} \frac{p' \cdot x}{p' \cdot x'}.$$

A strengthening of this condition turns out to be necessary and sufficient for rationalization by a homothetic preference. To simplify the notation, we shall normalize the prices in the data so that $p^k \cdot x^k = 1$ for all k. If we have prices so that $p^k \cdot x^k \neq 1$, it is simple to redefine prices $q^k = \frac{p^k}{p^k \cdot x^k}$ provided each $x^k \neq 0$. Theorem 4.2 will then apply to these normalized prices.

A dataset D satisfies the *homothetic axiom of revealed preference (HARP)* if, for any sequence $k_1, \ldots, k_L$ of numbers in $\{1, \ldots, K\}$, we have

$$p^{k_L} \cdot x^{k_1} \prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}} \geq 1.$$

Observe that HARP is stronger than GARP: Fix a sequence $k_1, \ldots, k_L$ such that GARP is violated; that is $p^{k_i} \cdot x^{k_{i+1}} \leq p^{k_i} \cdot x^{k_i}$, $i = 1, \ldots L-1$, and $p^{k_L} \cdot x^{k_1} < p^{k_L} \cdot x^{k_L}$. Then $p^{k_i} \cdot x^{k_i} = 1$ implies that the product of the left-hand side of these inequalities is strictly less than one. Thus HARP is violated.

Finally, let us define the *homothetic revealed preference order pair* $\langle \succeq^H, \succ^H \rangle$ as $x \succeq^H y$ when there are k and $\alpha > 0$ for which $\alpha x = x^k$ and $p^k \cdot x^k \geq p^k \cdot (\alpha y)$ and $x \succ^H y$ when there are k and $\alpha > 0$ for which $\alpha x = x^k$ and $p^k \cdot x^k > p^k \cdot (\alpha y)$. The meaning of $\succeq^H$ should be intuitive: $x \succeq^H y$ when the data, together with the hypothesis of homothetic preferences, imply that x must be preferred to y. Specifically, if x and x^k are on the same ray, then homotheticity requires that x be demanded at the same prices as x^k. This is like adding the observation (x, p^k) to the data. Then $p^k \cdot x > p^k \cdot y$ implies that x is "revealed preferred" to y, which is precisely the meaning of $x \succeq^H y$.

The following result is due to Hal Varian.

Theorem 4.2 *Let X be a convex cone such that for all $x \in X$ and all $\varepsilon > 0$, there is $\varepsilon' \in (0, \varepsilon)$ with $x + \varepsilon' \mathbf{1} \in X$. Let $D = \{(x^k, p^k)\}_{k=1}^K$ be a consumption dataset, where for all k, $p^k \cdot x^k = 1$. The following statements are equivalent:*

I) *D has a locally nonsatiated and homothetic weak rationalization.*
II) *D satisfies the homothetic axiom of revealed preference.*
III) *There is a strictly positive real number U^k for each k, such that*

$$U^k \leq U^l p^l \cdot x^k$$

for each pair of observations (x^k, p^k) and (x^l, p^l) in D.

IV) *D has a concave, homothetic, continuous, and monotonic rationalization* $u : X \to \mathbf{R}$.

V) $\langle \succeq^H, \succ^H \rangle$ *is acyclic.*

Proof. We first prove that I implies II. Let $\succeq$ be a locally nonsatiated, homothetic rationalization. Fix any sequence $k_1, \ldots, k_L$ in $\{1, \ldots, K\}$.

Let the sequence $s^1, \ldots, s^{L-1}$ be defined by

$$s^1 = p^{k_1} \cdot x^{k_2}$$
$$s^2 = (p^{k_1} \cdot x^{k_2}) p^{k_2} \cdot x^{k_3} = s^1 p^{k_2} \cdot x^{k_3}$$
$$\vdots$$
$$s^{L-1} = s^{L-2} p^{k_{L-1}} \cdot x^{k_L}.$$

We first argue that $x^{k_1} \succeq x^{k_2}/s^1$. Notice that the bundle x^{k_2}/s^1 is affordable in the budget at which x^{k_1} was purchased: $p^{k_1} \cdot (x^{k_2}/s^1) = 1$. Then $x^{k_1} \succeq (x^{k_2}/s^1)$. Next, we argue that homotheticity implies $(x^{k_2}/s^1) \succeq (x^{k_3}/s^2)$. The reason is that

$$p^{k_2} \cdot x^{k_2} = p^{k_2} \cdot \left(\frac{s^1}{s^2} x^{k_3} \right) = 1,$$

so that $x^{k_2} \succeq (s^1/s^2) x^{k_3}$, and hence by homotheticity, $(x^{k_2}/s^1) \succeq (x^{k_3}/s^2)$.

By repeating this argument we obtain that

$$x^{k_1} \succeq (x^{k_2}/s^1) \succeq (x^{k_3}/s^2) \succeq \cdots \succeq (x^{k_L}/s^{L-1}).$$

By assumption, x^{k_L} is $\succeq$ maximal in the set $\{x : p^{k_L} \cdot x \leq p^{k_L} \cdot x^{k_L}\}$, so x^{k_L}/s^{L-1} is $\succeq$ maximal in the set $\{x : s^{L-1} p^{k_L} \cdot x \leq s^{L-1} p^{k_L} \cdot x^{k_L}/s^{L-1}\}$ by the homotheticity of $\succeq$. Then $x^{k_1} \succeq (x^{k_L}/s^{L-1})$ and local nonsatiation of $\succeq$ implies $s^{L-1} p^{k_L} \cdot x^{k_1} \geq s^{L-1} p^{k_L} \cdot (x^{k_L}/s^{L-1})$. Since $s^{L-1} p^{k_L} \cdot (x^{k_L}/s^{L-1}) = 1$ we have established that HARP is satisfied.

Second, we prove that II implies III. Let the number U^l be defined as the infimum of

$$\prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}}$$

over all sequences $k_1, \ldots, k_L$ in $\{1, \ldots, K\}$ with $k_L = l$. The infimum is achieved for some sequence because HARP guarantees that removing a cycle only makes the product $\prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}}$ smaller, so we can without loss of generality consider the infimum over sequences with no cycles, and there are finitely many such sequences. Let $U^l = \prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}}$ and $U^m = \prod_{i=1}^{L'-1} p^{k'_i} \cdot x^{k'_{i+1}}$; then

$$U^l = \prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}} \leq (\prod_{i=1}^{L'-1} p^{k'_i} \cdot x^{k'_{i+1}})(p^m \cdot x^l) = U^m (p^m \cdot x^l).$$

Note that, following this construction, $U^k > 0$ for all k.

Third, we prove that III implies IV. Since we assumed that $p^l \cdot x^l = 1$ for all l, we have $U^l p^l \cdot x^k = U^l + U^l p^l \cdot (x^k - x^l)$ for all k, l, so that we have a solution to Afriat's inequalities. Define a utility function $u : X \to \mathbf{R}$ by

$$u(x) = \min\{U^k + U^k p^k \cdot (x - x^k) : k = 1, \ldots, K\},$$

a construction analogous to the one in the proof of Afriat's Theorem. Then $p^k \cdot x^k \geq p^k \cdot y$ implies that $u(x^k) \geq u(y)$, so u rationalizes the data.

Clearly, u is continuous, monotonic, and homothetic. The proof that it is concave is the same as in Afriat's Theorem.

Finally, we demonstrate that (II) and (V) are equivalent. Suppose by means of contradiction that there is a cycle $z^1 \succeq^H \ldots \succeq^H z^L \succ^H z^1$. For each $i = 1, \ldots, L$, suppose $\alpha^i z^i = x^{k_i}$. Now, $z^i \succeq^H z^{i+1}$ means that $p^{k_i} \cdot x^{k_i} \geq p^{k_i} \cdot \left(\alpha^i z^{i+1}\right) = p^{k_i} \cdot \left(\frac{\alpha^{k_i} x^{k_{i+1}}}{\alpha^{k_{i+1}}}\right)$. Then for all $i = 1, \ldots, L-1$, $1 = p^{k_i} \cdot x^{k_i} \geq p^{k_i} \cdot \left(\frac{\alpha^{k_i} x^{k_{i+1}}}{\alpha^{k_{i+1}}}\right)$, and $1 = p^{k_L} \cdot x^{k_L} > p^{k_L} \cdot \left(\frac{\alpha^{k_L} x^{k_1}}{\alpha^{k_1}}\right)$. Multiplying the inequalities and canceling the α terms obtains $p^{k_L} \cdot x^{k_1} \prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}} < 1$, violating HARP.

On the other hand, suppose (II) is violated, and let $k_1, \ldots, k_L$ be a sequence for which $p^{k_L} \cdot x^{k_1} \prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}} < 1$. Define $z^1 = x^{k_1}$, $\alpha^1 = 1$, and for each $i = 2, \ldots, L$, define $\alpha^i = p^{k_{i-1}} \cdot x^{k_i}$, and $z^i = \frac{x^{k_i}}{\prod_{j=1}^{i} \alpha^j}$. Observe that for all $i = 1, \ldots, L-1$, $z^i \succeq^H z^{i+1}$, since $p^{k_i} \cdot (\alpha^i z^i) = p^{k_i} \cdot (\alpha^i z^{i+1})$. Observe also that $\left(\prod_{i=1}^{L} \alpha^L\right) p^{k_L} \cdot z^L = p^{k_L} \cdot x^{k_L} = 1$, and that $\left(\prod_{i=1}^{L} \alpha^L\right) p^{k_L} \cdot z^1 = p^{k_L} \cdot x^{k_1} \prod_{i=1}^{L-1} p^{k_i} \cdot x^{k_{i+1}} < 1$ by assumption, so that $z^L \succ^H z^1$, constituting a $\langle \succeq^H, \succ^H \rangle$ cycle.

Remark 4.3 The preceding does not allow observations of $x^k = 0$. If we did, the "non-normalized" version of HARP and condition (III) are necessary and sufficient here. The non-normalized version of HARP would read that for all sequences $\{k_1, \ldots, k_n\}$, we have $p^{k_n} \cdot x^{k_1} \prod_{i=1}^{n-1} p^{k_1} \cdot x^{k_{i+1}} \geq \prod_{i=1}^{n} p^{k_i} \cdot x^{k_i}$. The non-normalized version of condition (III) is that $U^k p^l \cdot x^l \leq U^l p^l \cdot x^k$ for all k and l.

4.2.2 Separability

In practical analysis of consumer demand, separability is a very important property. In principle, a consumer chooses among many different goods, solving inter- as well as intratemporal optimization problems. Practical researchers abstract away from this complexity by considering some subset of goods in isolation. For example, classical studies of applied demand, such as Deaton (1974), work with only 9 goods. Modern applied papers often consider more goods, but still greatly simplify the universe of possible goods.

The simplification of focusing on a small subset of goods and assuming separability avoids the issues raised in Section 3.2.3. In fact, assuming separability of some kind seems unavoidable for any tractable empirical study of consumption.

Let $X \subseteq X_1 \times X_2$, with $X_i \subseteq \mathbf{R}^{n_i}_+$; and write vectors in X as (x_1,x_2), where $x_i \in X_i$. A preference relation $\succeq$ is *separable* in X_1 if for all $x_1,x_1' \in X_1$ and all $x_2,\hat{x}_2 \in X_2$ $(x_1,x_2) \succeq (x_1',x_2)$ if and only if $(x_1,\hat{x}_2) \succeq (x_1',\hat{x}_2)$. A utility function $u : X \to \mathbf{R}$ is *separable* in X_1 if there are functions $f : X_1 \to \mathbf{R}$ and $g : f(X_1) \times X_2 \to \mathbf{R}$, where for all x_2, $y \mapsto g(y,x_2)$ is strictly monotonic, such that $u(x_1,x_2) = g(f(x_1),x_2)$.

When preferences are separable, the consumer's maximization problem reduces into a "subproblem" for the goods in X_1. Given a budget to spend on goods x_1, the consumer can solve the problem of choosing x_1 independently of the specific bundle x_2 chosen: only the budget left over to spend on x_1 matters, not the actual quantities of the goods in x_2. Thus separability allows the separate analysis of demand for x_1. Of course, here we are concerned with testing for separability, so we cannot ignore the relation between choosing x_1 and x_2.

Theorem 4.4 *The following statements are equivalent:*

I) *There are strictly positive real numbers U^k, V^k, λ^k, μ^k, $k = 1,\dots,K$ such that*

$$U^k \leq U^l + \lambda^l p_1^l \cdot (x_1^k - x_1^l) + \frac{\lambda^l}{\mu^l}(V^k - V^l), \tag{4.1}$$

$$V^k \leq V^l + \mu^l p_2^l \cdot (x_2^k - x_2^l). \tag{4.2}$$

II) *The dataset $\{(p_2^k,x_2^k)\}$ satisfies GARP; and there is a solution (V^k,μ^k) to 4.2 above such that the dataset $\{((p_1^k,1/\mu^k),(x_1^k,V^k)\}$ satisfies GARP.*

III) *D has a concave, continuous, and monotonic rationalization $u : X \to \mathbf{R}$ that is separable in X_1.*

We present Theorem 4.4 without proof. It is important to note that Theorem 4.4 focuses on concave rationalizations (note the contrast with Afriat's Theorem, where concavity comes for free). The Afriat inequalities for this problem are described by Equations (4.1) and (4.2). These inequalities are not linear, in contrast with the original Afriat inequalities.

A special kind of separability is *additive separability*. A test for additive separability is particularly interesting for time-series data (see Section 5.3.3), in which there is a single observation (a dataset of size 1).

Let $X \subseteq X_1 \times X_2 \times \cdots \times X_T$, with $X_t = \mathbf{R}^n_+$, $t = 1,\dots,T$. Write vectors in X as $(x_1,\dots,x_T)$, where $x_t \in \mathbf{R}^n_+$. A preference relation $\succeq$ is *additively separable* if there is a function $u : \mathbf{R}^n_+ \to \mathbf{R}$ such that $(x_1,\dots,x_T) \succeq (x_1',\dots,x_T')$ iff $\sum_{t=0}^T u(x_t) \geq \sum_{t=0}^T u(x_t')$.

Say that a dataset $\{(x^k,p^k)\}_{k=1}^K$ is *additively separably rationalizable* by the function $u : \mathbf{R}^n_+ \to \mathbf{R}$ if the additively separable preferences defined from u weakly rationalize the data $\{(x^k,p^k)\}_{k=1}^K$. We provide a test for the case when $K = 1$, so the dataset is (x,p). Note that $p = (p_1,\dots,p_T)$, with $p_t \in \mathbf{R}^n_{++}$.

Proposition 4.5 *Data (x,p) (a dataset with $K = 1$) with $x \gg 0$ is additively separably rationalizable by a concave and strictly increasing function iff the correspondence $\rho(x) = \bigcup_{t:x=x_t}\{p_t\}$ satisfies cyclic monotonicity.*

Proposition 4.5 follows from using the first-order conditions for maximization of $\sum_t u(x_t)$, and Corollary 1.10.

Note that general utility maximization is not testable with a single observation, but the theory of additive separability has testable implications even when $K = 1$.

4.2.3 Quasilinear utility

Many economic models assume that utility takes a quasilinear form: $u(x)+y$, for $x \in \mathbf{R}^n_+$ and $y \in \mathbf{R}$. There are $n+1$ goods, and the $(n+1)$-st is a *numeraire*. Wealth is measured in the same units as the numeraire good, which therefore always has a price of one. We shall allow consumption of the numeraire to be negative, which is a common assumption in applications of quasilinear utility. Given the assumption that y can be negative, the maximization of $u(x)+y$ subject to a budget constraint with prices p (the problem $\max_{(x,y)} u(x)+y$ subject to the constraint $p \cdot x + y \leq I$) is, when u is locally nonsatiated, equivalent to the maximization of $u(x) - p \cdot x$.

As in Proposition 4.5, the property of cyclic monotonicity described in 1.4 can be used to characterize datasets that could be rationalized by a quasilinear utility.

We take as primitive a dataset D: a collection (x^k,p^k), $k = 1,\ldots K$, with $K \geq 1$ an integer, $x^k \in X$ and $p^k \in \mathbf{R}^n_{++}$. There are now, however, $n+1$ goods and we seek a rationalization by a quasilinear utility function. A dataset D is *quasilinear rationalizable* if there exists a locally nonsatiated utility function $u : X \to \mathbf{R}$ such that for all k and $x \in X$,

$$u(x) - p^k \cdot x \leq u(x^k) - p^k \cdot x^k.$$

Theorem 4.6 *Let X be a convex consumption space such that for all $x \in X$ and all $\varepsilon > 0$, there is $\varepsilon' \in (0,\varepsilon)$ with $x + \varepsilon'\mathbf{1} \in X$. The following statements are equivalent:*

I) *D is quasilinear rationalizable.*
II) *For each k, there is U^k such that for all k,l,*

$$U^k \leq U^l + p^l \cdot (x^k - x^l).$$

III) *The correspondence $\rho(x) = \bigcup_{k:x=x^k}\{p^k\}$ satisfies cyclic monotonicity.*
IV) *The data are quasilinear rationalizable by a continuous, strictly increasing, concave utility function.*

Proof. That (I) implies (II) is as follows. Let u rationalize the data, and let $U^k = u(x^k)$. In particular, we have $U^l - p^k \cdot x^l \leq U^k - p^k \cdot x^k$, which establishes II.

That (II) implies (III) follows from adding up Afriat inequalities corresponding to cycles, as we discussed after stating Afriat's Theorem (and by the same argument as in the proof of Theorem 1.9). The proof that (III) implies (IV) is exactly as in Theorem (1.9): let

$$u(x) = \inf\{p^{k_1} \cdot (x - x^{k_1}) + p^{k_2} \cdot (x^{k_1} - x^{k_2}) + \ldots + p^{k_{M-1}} \cdot (x^{k_M} - x^{k_{M-1}})\},$$

where the infimum is taken over all sequences $k_1, \ldots, k_M$ with x^{k_1} fixed at some arbitrary $x_0 \in X$. (See Corollary 1.10).

That (IV) implies (I) is trivial.

Remark 4.7 If one assumes that income and quasilinear good consumption must be positive, then the equivalence of (II)–(IV) in the preceding still hold. That (IV) implies (II) holds from the first-order conditions of concave optimization. Condition (IV) could be proved by choosing I large enough so that $I - p^i \cdot x^i \geq 0$ for all i, and defining $y^i = I - p^i \cdot x^i$.

The idea behind Theorem 4.6 is similar to the argument in Theorem 1.9, and it is instructive to see why. If we reason as in the discussion of Afriat's Theorem, and assume that there is a differentiable rationalization u, then the relevant first-order condition for maximizing $u(x) - p \cdot x$ is

$$\nabla u(x) = p.$$

As was the case for Afriat's Theorem, we seek to infer marginal utilities from data, but the difference is that now the Lagrange multiplier (λ^k in Afriat's Theorem) is known and equal to one. The dataset therefore already tells us what the marginal utilities must be, if the data are to be rationalized by a quasilinear utility. Theorem 4.6 therefore asks for prices that could be marginal utilities for a concave utility function – namely cyclic monotonicity. Most of the work in proving Afriat's Theorem went into establishing the existence of multipliers λ^k such that $\rho(x) = \bigcup_{k:x=x^k}\{\lambda^k p^k\}$ is cyclically monotone. In the case of quasilinear utility, this is already taken care of by the quasilinearity assumption.

4.2.4 Gross complements and substitutes

We now turn to two basic properties of demand: the complementarity or substitutability between a pair of goods. Roughly speaking, two goods are "gross" complements if a price change that favors the consumption of one good also induces higher consumption of the second good. Common examples of complementary goods include coffee and sugar, or gin and tonic. Instead of being properties of preference, gross complements and gross substitutes are properties of demand functions.

Formally, we say that a demand function on $\mathbf{R}^2_+$ satisfies *gross complements* if for all m, $d(p,m)$ is a weakly decreasing function of p. We say that the demand function is *rational* if there exists a locally nonsatiated preference relation $\succeq$ for which there is a unique $\succeq$-maximal element of $\{x \in \mathbf{R}^n_+ : p \cdot x \leq m\}$, and $d(p,m)$ is equal to this element. A demand function satisfying gross complements is one for which both goods respond in the same direction to changes in price. On the other hand, we will say a demand function on $\mathbf{R}^2_+$ satisfies *gross substitutes* if $d_1(p,m)$ increases in p_2, and $d_2(p,m)$ increases in p_1. Note that for these definitions to be at all meaningful, demand must be single-valued. We say a demand function is a *rational demand function* if there is a monotonic preference which generates d as its demand correspondence.

It is convenient here to suppose that m is normalized; this is without loss of generality as any rational demand function is homogeneous. For the remainder of this subsection, we always assume $m = 1$, and drop the dependence of d on m.

Data come in the form of observed price/demand pairs, $D = \{(p^k, x^k)\}_{k=1}^K$. We turn to the question of when data are consistent with a rational demand function exhibiting the property of gross complements. In line with the normalization of income, assume that data has been normalized such that, for all k, $p^k \cdot x^k = 1$. Consider Figure 4.3(a), which depicts a hypothetical observation of demand $x = (x_1, x_2)$ at prices $p = (p_1, p_2)$. In principle, the two budgets in Figure 4.3(a) are not comparable, and the observations might be consistent with gross complements. However, the dotted budget line in the figure can be obtained by either starting from (x, p) and making one good cheaper, or by starting from (x', p') and making the other good cheaper. Either way, demand at the dotted budget line should be larger than both x and x'. As Figure 4.3(b) illustrates, this is not possible. In this fashion we obtain a condition on the data that is necessary for consistency with gross complements: the pointwise maximum of demands, $x \vee x'$, must be affordable for any budget larger than the p and p' budgets.

There is a second necessary condition. Consider the observed demands in Figure 4.3(c). This a situation where, when we go from p to p', demand for the good that gets cheaper decreases while demand for the good that gets more expensive increases. This is not in itself a violation of complementarity. However, consider Figure 4.3(d): were we to increase the budget from p to the dotted prices, complementarity would imply a demand at the dotted prices that is larger than x. But no point in the dotted budget line is both larger than x and satisfies the weak axiom of revealed preference (WARP) with respect to the choice of x'.

So a simultaneous increase in one price and decrease in another cannot yield opposite changes in demand. This property is a strengthening of WARP: Fix p, p' and x as in Figure 4.3(c). Then WARP requires that x' not lie below the point where the p and p' budget lines cross. Our property requires that x' not lie below the point on the p'-budget line with the same quantity of good 2 as x.

The following result states that the two necessary conditions illustrated in Figure 4.3 are in fact both necessary and sufficient. These constitute a nonparametric test of gross complements in the demand for a pair of goods.

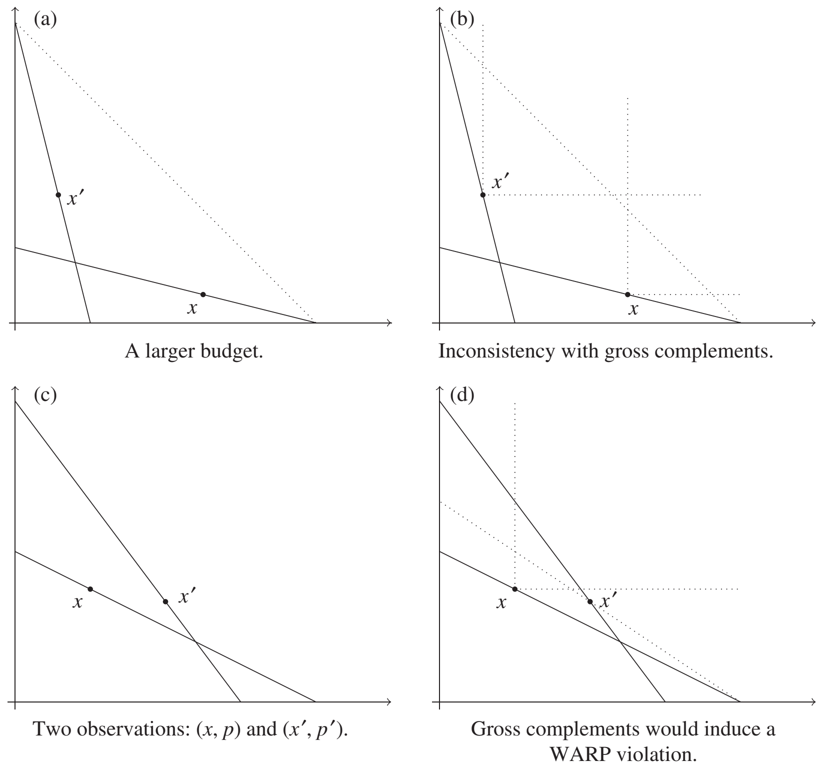
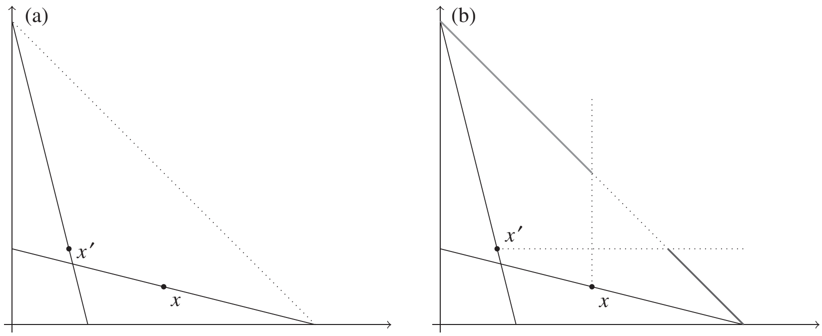


Fig. 4.3 Implications of gross complements.

Theorem 4.8 *Let $\{(p^k,x^k)\}_{k=1}^K$ be a dataset with $n = 2$ and, for all k, $p^k \cdot x^k = 1$ (a normalization). There exists a rational demand d, satisfying gross complements, such that for all k, $d(p^k) = x^k$ iff for all k,l, the following are satisfied:*

- $(p^k \wedge p^l) \cdot (x^k \vee x^l) \leq 1$
- *For all $i \neq j$, if $p^k \cdot x^l \leq 1$ and $p^k_i > p^l_i$, then $x^k_j \geq x^l_j$.*

We now turn to gross substitutes. To illustrate the implications of gross substitutes for observed demand, consider the example in Figure 4.4. We have two observations: x is the bundle purchased at prices p, and x' is purchased at prices p'. These purchases do not appear to directly violate gross substitutes. The observed choices are also consistent with the weak axiom of revealed preference, so there is an extension of these purchases to a rational demand function that is defined for all prices. There is, however, no demand function compatible with these observations which satisfies gross substitutes: Consider the prices p'' given by the dotted budget line. Gross substitutes and the choice of x at p require a decrease in the consumption of the good whose price is the same in p and in p'', so demand at p'' should lie in the northwest segment of the



Two observations: (x, p) and (x', p'). Implications of gross substitutes.

Fig. 4.4 Implications of gross substitutes.

budget line. On the other hand, gross substitutes and the choice of x' require that demand at p'' lies in the southeast segment of the budget line. Since these two segments are disjoint, there is no demand function that extends the data and satisfies gross substitutes.

Figure 4.4 then suggests a necessary condition for observations to be consistent with a rational demand exhibiting gross substitutes. The next theorem states that the condition is both necessary and sufficient. Smoothness of demand comes for free in this case.

Theorem 4.9 *Let $\{(p^k, x^k)\}_{k=1}^K$ be a dataset with $n = 2$ and, for all k, $p^k \cdot x^k = 1$ (a normalization). There exists a smooth and rational demand function d satisfying gross substitutes such that for all k, $d(p^k) = x^k$ iff for all k, l such that $p_1^k \leq p_1^l$ and $p_2^l \leq p_2^k$, we have $p_1^l x_1^l \leq p_1^k x_1^k$.*

4.3 CHAPTER REFERENCES

A version of Theorem 4.1 appeared first in Chambers and Echenique (2009b), and was extended by Shirai (2010). Our proof of the supermodular result follows suggestions by Eran Shmaya and John Quah. The submodular utility used in the proof is a construction due to Rader (1963) (Shirai noted that this construction yields a submodular utility).

The results in Sections 4.2.1 and 4.2.2 are from Varian (1983a). The equivalence between conditions (III) and (I) in Section 4.2.1 was shown by Afriat (1972) and Diewert (1973); HARP is due to Varian, though Diewert mentions an equivalent test in a footnote. Knoblauch (1993) describes a method for predicting responses to price changes for data consistent with homotheticity.

Theorem 4.4 is due to Hal Varian. The recent paper of Quah (2013) is the first paper to treat the case of a general, possibly nonconcave, separable rationalization. In fact Quah presents an example of a dataset that has a separable rationalization, but not a concave and separable one. Quah shows that to test for separability one needs to verify that data satisfy some finite set of configurations. The paper by Cherchye, Demuynck, Hjertstrand, and De Rock (2014) shows that testing for separability in Varian's concave setting is computationally hard, and provides a computational approach to dealing with the system of inequalities in Theorem 4.4. Echenique (2013) proves that the nonconcave case is also computationally hard, even when the number of goods is as small as in Deaton (1974) (i.e. 9 goods).

Proposition 4.5 appears in Browning (1989), who uses it to test for additive separability in time series data. He focuses on the case of a single observation because households in consumption surveys only make a single choice. The paper of Echenique, Imai, and Saito (2013) presents a result on time separability for multiple observations (as well as for other models of intertemporal choice).

The result on quasilinear utility is from Brown and Calsamiglia (2007).

Theorem 4.8 is due to Chambers, Echenique, and Shmaya (2010), while Theorem 4.9 appears in Chambers, Echenique, and Shmaya (2011). Related are the papers of Kehoe and Mas-Colell (1984) and Kehoe (1992), which show that gross substitutability of demand implies a version of the weak axiom.