Production

Production theory is another classical environment in which revealed preference theory is applied. The case of production is simpler than the case of demand treated in the previous chapters, mainly because firm output is a cardinally measurable and observable concept, whereas utility is not. In the case of production, we shall assume that firm output and prices are both observed, while the set of all feasible production vectors, that is the firm's technology, is not.

We will consider two approaches to production theory: the cost minimization model and the profit maximization model. In the first model, factor prices and factor demands are observed, and (single-dimensional) output is observed as well. This environment is very similar to the consumer case, but, as we have noted, simpler. We want to know whether the model is consistent with the cost minimization hypothesis, meaning that the cost of production is minimized for a given level of output.

In the second model, the model of profit maximization, we want to test the hypothesis that producers maximize profit. This model is in a sense "dual" to the consumer case. In the consumer case, we needed to solve for the function being maximized, but we know the budget set. In contrast, in the producer case, we know the function being maximized: it is a linear profit function; but we do not necessarily know the available technology (the constraint set faced by the firm).

6.1 COST MINIMIZATION

We take as primitive a dataset comprising the input–output decisions of a firm. The firm uses n factors, and produces a single good. An *input–output dataset* D consists of a collection (y^k, x^k, p^k), $k = 1, \ldots, K$, where $y^k \in \mathbf{R}$, $x^k \in \mathbf{R}^n_+$, and $p^k \in \mathbf{R}^n_{++}$. Each observation k consists of a quantity of output y^k, a vector of factor demands x^k, and factor prices p^k. Note that output can be negative, but inputs are always positive.

A *production function* is a mapping $f : \mathbf{R}^n_+ \to \mathbf{R}$. We say that a production function f *cost rationalizes* input–output dataset D if $f(x^k) = y^k$ for all k, and

$f(x) \geq f(x^k)$ implies that $p^k \cdot x \geq p^k \cdot x^k$. In other words, given prices p^k, x^k is a cost-minimizing bundle across all bundles which can, according to f, produce at least y^k.

Theorem 6.1 *The following statements are equivalent:*

I) *There is a continuous production function that cost rationalizes D.*
II) $y^j \leq y^k$ *implies* $p^j \cdot x^j \leq p^j \cdot x^k$ *and* $y^j < y^k$ *implies* $p^j \cdot x^j < p^j \cdot x^k$.
III) *There are real numbers $U^k, \lambda^k > 0$ for which $y^j \leq y^k$ implies $U^j \leq U^k$ and $y^j < y^k$ implies $U^j < U^k$, and for all j,k,*

$$U^j \leq U^k + \lambda^k p^k \cdot (x^j - x^k).$$

IV) *There is a continuous, monotonic, and quasiconcave production function that cost rationalizes D.*

Equation (II) plays the role of GARP in this context. If we were to define the revealed preference by $x^k \succeq^R x^j$ if $p^k \cdot x^j \leq p^k \cdot x^k$, and $x^k \succ^R x^j$ if $p^k \cdot x^j < p^k \cdot x^k$, then (II) guarantees that $\langle \succeq^R, \succ^R \rangle$ is acyclic.

The equivalence between (III) and (IV) in Theorem 6.1 is reminiscent of Afriat's Theorem. The statement in (III) gives the "Afriat inequalities" corresponding to the problem under consideration. Note, however, that (III) says more than in Afriat's Theorem. The reason is that we observe production output, the analogue of utility in demand theory, while utility is not observable.

Proof. To see that (I) implies (II), we first note that if $y^j \leq y^k$ then since f cost rationalizes D, we have $p^j \cdot x^j \leq p^j \cdot x^k$. If $y^j < y^k$, then we know that $p^j \cdot x^j \leq p^j \cdot x^k$, and if in fact $p^j \cdot x^j = p^j \cdot x^k$, we cannot have $x^k = 0$, as otherwise, $x^k = x^j = 0$, which would imply $y^k = f(x^k) = f(x^j) = y^j$, a contradiction. Since $f(x^k) = y^k > y^j = f(x^j)$, and f is continuous, there is $x < x^k$ for which $f(x) > f(x^j)$, yet $p^j \cdot x < p^j \cdot x^j$, a contradiction to the fact that f cost rationalizes D.

To see that (II) implies (III), we refer to Afriat's Theorem. We may define the revealed preference relations $\succeq^R$ and $\succ^R$ in the same way they are defined in Chapter 3, so that $x^j \succeq^R x^k$ if $p^j \cdot x^k \leq p^j \cdot x^j$, and $x^j \succ^R x^k$ if $p^j \cdot x^k < p^j \cdot x^j$.

By (II), $x^j \succeq^R x^k$ implies $y^j \geq y^k$ and $x^j \succ^R x^k$ implies $y^j > y^k$. It follows that the preference relation $\succeq$ on $X^0 = \{x^k : k = 1, \ldots, K\}$ defined by $x^j \succeq x^k$ iff $y^j \geq y^k$ is such that $x^j \succeq^R x^k$ implies $x^j \succeq x^k$ and $x^j \succ^R x^k$ implies $x^j \succ x^k$. The result then follows by the constructive proof of Afriat's Theorem, so that the desired numbers exist. It is easily verified that $y^j \leq y^k$ implies $U^j \leq U^k$ and $y^j < y^k$ implies $U^j < U^k$.

Finally, to see that (III) implies (IV), let $u : \mathbf{R}^n_+ \to \mathbf{R}$ be the utility function as constructed in Afriat's Theorem. Recall that u is strictly increasing and concave and that $u(x^k) = U^k$. Observe first that if $u(x) \geq u(x^k)$, then it follows that $U^k + \lambda^k p^k \cdot (x - x^k) \geq u(x) \geq U^k$, so that $p^k \cdot x^k \leq p^k \cdot x$.

We now let φ be any strictly increasing transformation of u for which $\varphi(u(x^k)) = y^k$ (that this is possible follows as $y^j \leq y^k$ implies $u^j \leq u^k$ and $y^j < y^k$ implies $u^j < u^k$). Then let $f = \varphi \circ u$, and note that f is strictly increasing and quasiconcave. Finally, it cost rationalizes the data: $f(x) \geq f(x^k)$ implies that $u(x) \geq u(x^k)$, which we have shown implies $p^k \cdot x^k \leq p^k \cdot x$.

Theorem 6.1 describes the datasets that are rationalizable by a quasiconcave production function. Unlike the demand context, concavity here will impose additional testable restrictions. For example, consider an environment with one input. Suppose three observations are given in D: $(y^1, x^1, p^1) = (0,0,1)$, $(y^2, x^2, p^1) = (1,1,1)$, and $(y^3, x^3, p^3) = (3,2,1)$. Note that D is cost rationalizable by a quasiconcave production function; for example, $f(x) = \max\{x, 2x-1\}$ cost rationalizes the data. However, any f which cost rationalizes D must satisfy $f(0) = 0$, $f(1) = 1$, and $f(2) = 3$. No such function can be concave.

To this end, we can also test when an input–output dataset can be cost rationalized by a *concave* production function.

Theorem 6.2 *The following are equivalent:*

I) *There is a concave production function that cost rationalizes D.*
II) *For each k, and all $\alpha^j \geq 0$ for which $\sum_{j \neq k} \alpha^j = 1$, if $p^k \cdot (\sum_{j \neq k} \alpha^j x^j) \leq p^k \cdot x^k$, then $\sum_{j \neq k} \alpha^j y^j \leq y^k$, and if $p^k \cdot (\sum_{j \neq k} \alpha^j x^j) < p^k \cdot x^k$, then $\sum_{j \neq k} \alpha^j y^j < y^k$.*
III) *There is a concave, monotonic, and continuous production function that cost rationalizes D.*

Proof. We first show that (I) implies (II). Suppose D can be cost rationalized by a concave production function f, and suppose that $p^k \cdot x^k \geq p^k \cdot \left(\sum_{j \neq k} \alpha^j x^j\right)$ for some α^j as in statement (II). Then suppose by way of contradiction that $\sum_{j \neq k} \alpha^j y^j > y^k$. In particular this implies that $\sum_{j \neq k} \alpha^j f(x^j) > f(x^k)$. By concavity, we know that $f\left(\sum_{j \neq k} \alpha^j x^j\right) \geq \sum_{j \neq k} \alpha^j f(x^j) > f(x^k)$. First, we show that $\sum_{j \neq k} \alpha^j x^j \neq 0$. Suppose, toward a contradiction, that $\sum_{j \neq k} \alpha^j x^j = 0$. Choose some j for which $\alpha^j > 0$ and $y^j > y^k$. We must have $x^j = 0$, as $\sum_{j \neq k} \alpha^j x^j = 0$, which allows us to conclude that $y^j = f(0) > f(x^k)$. By cost rationalization, we then have that $0 = p^k \cdot 0 \geq p^k \cdot x^k$, which implies that $x^k = 0$, contradicting $f(0) > f(x^k)$. This shows that $\sum_{j \neq k} \alpha^j x^j > 0$.

Next, on the interior of any one-dimensional subset of $\mathbf{R}^n$, f is continuous (as it is concave). In particular, except possibly at the origin, f is continuous on the ray passing through the origin and $\sum_{j \neq k} \alpha^j x^j$. So, consider $\beta < 1$ for which $f\left(\beta\left(\sum_{j \neq k} \alpha^j x^j\right)\right) > f(x^k)$. Then since $p^k \in \mathbf{R}^n_{++}$, we know that $p^k \cdot \left(\beta\left(\sum_{j \neq k} \alpha^j x^j\right)\right) < p^k \cdot x^k$, a contradiction to the cost rationalization hypothesis. This establishes that $\sum_{j \neq k} \alpha^j y^j \leq y^k$.

To complete the proof of (II), suppose that that $p^k \cdot x^k > p^k \cdot \left(\sum_{j \neq k} \alpha^j x^j\right)$. Suppose, toward a contradiction, that $\sum_{j \neq k} \alpha^j y^j \geq y^k$. Then $f(x^k) \leq \sum_{j \neq k} \alpha^j f(x^j) \leq f\left(\sum_{j \neq k} \alpha^j x^j\right)$, where the inequality follows by concavity. This is a direct contradiction to the cost rationalization hypothesis, as $\sum_{j\neq k}\alpha^j x^j$ can produce y^k at a lower cost than x^k.

Second, we prove that (II) implies (III). To that end, we show the existence of $\lambda^k > 0$ such that for all j,k,

$$y^k \leq y^j + \lambda^j p^j \cdot (x^k - x^j). \tag{6.1}$$

The existence of such λ^k is equivalent to the existence of $\lambda^k > 0$ and $\mu > 0$ such that for all j,k,

$$\mu(y^j - y^k) + \lambda^j p^j \cdot (x^k - x^j) \geq 0.$$

If we can solve these inequalities, we can renormalize, setting $\mu = 1$, and obtain a solution to (6.1). Now, by Lemma 1.12, there is no solution exactly when, for each (j,k) where $j \neq k$, there is $\alpha^{(j,k)} \geq 0$ for which $\sum_{(j,k)}\alpha^{(j,k)}(y^j - y^k) \leq 0$ and for all j, $\sum_{k\neq j}\alpha^{(j,k)}p^j \cdot (x^k - x^j) \leq 0$, and at least one of these inequalities is strict. Since for all j, $\sum_{k\neq j}\alpha^{(j,k)}p^j \cdot (x^k - x^j) \leq 0$, we get by (II) that for every j, $\sum_{k\neq j}\alpha^{(j,k)}(y^k - y^j) \leq 0$, with a strict inequality if the original inequality is strict. By summing across j, we have $\sum_{(j,k)}\alpha^{(j,k)}(y^k - y^j) \leq 0$, with a strict inequality if any of the inequalities corresponding to some j is strict, a contradiction.

The construction of a production function is the same as in Afriat's Theorem. Let $f(x) = \min_k y^k + \lambda^k p^k \cdot (x - x^k)$, a concave, continuous, and strictly monotonic function. Cost rationalization is verified as in Theorem 6.1. Finally, for all k, $f(x^k) = y^k$.

Often we want to ensure that $f(x) \geq 0$ for all x. A test for this is provided by weakening the equality $\sum_{j\neq k}\alpha^j = 1$ in Theorem 6.2.

Theorem 6.3 *The following are equivalent:*

I) *There is a non-negative concave production function that cost rationalizes D.*
II) *For each k, and all $\alpha^j \geq 0$ for which $\sum_{j\neq k}\alpha^j \leq 1$, if $p^k \cdot (\sum_{j\neq k}\alpha^j x^j) \leq p^k \cdot x^k$, then $\sum_{j\neq k}\alpha^j y^j \leq y^k$, and if $p^k \cdot (\sum_{j\neq k}\alpha^j x^j) < p^k \cdot x^k$, then $\sum_{j\neq k}\alpha^j y^j < y^k$.*
III) *There is a non-negative, concave, monotonic, and continuous production function that cost rationalizes D.*

Proof. We first establish that (I) implies (II). The only difference from the proof of Theorem 6.2 is that the equation $f\left(\sum_{j\neq k}\alpha^j x^j\right) > f(x^k)$ is established by observing that $f\left(\sum_{j\neq k}\alpha^j x^j\right) = f\left((1 - \sum_{j\neq k}\alpha^j x^j)0 + \sum_{j\neq k}\alpha^j x^j\right) \geq (1 - \sum_{j\neq k}\alpha^j)0 + \sum_{j\neq k}\alpha^j f(x^j)$ follows from non-negativity and concavity instead of concavity alone.

To see that (II) implies (III), we add, for every j, an inequality of the form: $y^j + \lambda^j p^j \cdot (-x^j) \geq 0$ to the list of inequalities described in Theorem 6.2. Equivalently, in terms of the expression involving μ, this adds, for every j, an inequality of the form:

$$\mu y^j + \lambda^j p^j \cdot (-x^j) \geq 0.$$

Again, by Lemma 1.12, there is no solution exactly when, for each j, there is η^j and for each (j,k) where $j \neq k$, there is $\alpha^{(j,k)} \geq 0$ for which $\left(\sum_j \eta^j y^j\right) + \left(\sum_{(j,k)} \alpha^{(j,k)}(y^j - y^k)\right) \leq 0$ and for all j, $\eta^j p^j \cdot (-x^j) + \sum_{k \neq j} \alpha^{(j,k)} p^j \cdot (x^k - x^j) \leq 0$, and at least one of these inequalities is strict. Since for all j, $\eta^j p^j \cdot (-x^j) + \sum_{k \neq j} \alpha^{(j,k)} p^j \cdot (x^k - x^j) \leq 0$, we get by (II) that for every j, $\eta^j(-y^j) + \sum_{k \neq j} \alpha^{(j,k)}(y^k - y^j) \leq 0$, with a strict inequality if the original inequality is strict. By summing across j, we have $\left(\sum_j \eta^j\right) + \left(y^j \sum_{(j,k)} \alpha^{(j,k)}(y^k - y^j)\right) \leq 0$, with a strict inequality if any of the inequalities corresponding to some j is strict, a contradiction.

The construction of f used in the proof of Theorem 6.2 ensures non-negativity; indeed, observe that $f(0) = \min_k y^k + \lambda^k p^k \cdot (-x^k) \geq 0$. Non-negativity then follows from monotonicity.

6.2 PROFIT MAXIMIZATION

The previous section assumed a firm with a single output y using n factors of production. We now turn to a more flexible formulation in which a firm operates in n goods, choosing a net production vector $y \in \mathbf{R}^n$. If $y_i > 0$ then good i is produced in quantity y_i by the firm. If $y_i < 0$ then the firm uses good i as an input.

A *production dataset* D is a collection (y^k, p^k), $k = 1, \ldots K$, with $y^k \in \mathbf{R}^n$ and $p^k \in \mathbf{R}^n_{++}$.

We are interested in when a production dataset D is consistent with the hypothesis of profit maximization. A *production set* Y is a subset of $\mathbf{R}^n$. Production sets consist of all potential combinations of inputs and outputs which are feasible. We say that production set Y *rationalizes* production dataset D if for all k, $y^k \in Y$ and $p^k \cdot y^k \geq p^k \cdot y$ for all $y \in Y$. We say a production dataset D is rationalizable if it is rationalizable by a production set. We will say that Y is *comprehensive* if whenever $y \in Y$ and $y' \leq y$, then $y' \in Y$.

Given is a production dataset D.

Theorem 6.4 *The following statements are equivalent:*

I) *For all j,k, $p^k \cdot y^k \geq p^k \cdot y^j$.*
II) *D is rationalizable.*
III) *D is rationalizable by a closed, convex, and comprehensive production set.*

Proof. That (III) implies (II) and (II) implies (I) are obvious. To see that (I) implies (III), let Y be the convex and comprehensive hull of $\{y^k\}_{k=1}^K$.[1] Y is obviously closed, convex, and comprehensive. We claim that Y rationalizes D. To see this, it is first clear that $y^k \in Y$ for all k, by definition. Second, suppose that $y \in Y$. We want to show that for any j, $p^j \cdot y^k \geq p^k \cdot y$. By definition of the convex and comprehensive hull, there exists $y' = \sum_{k=1}^K \lambda_k y^k$, where $\lambda_k \geq 0$ for all k and $\sum_{k=1}^K \lambda_k = 1$, where $y \leq y'$ (this is an easy set-theoretic argument). Consequently,

$$p^j \cdot y \leq p^j \cdot y' = \sum_{k=1}^K \lambda_k (p^j \cdot y^k) \leq \sum_{k=1}^K \lambda_k (p^j \cdot y^j) = p^j \cdot y^j,$$

where the last inequality follows from (I).

It is usually assumed that $0 \in Y$, because a firm can always choose to do nothing. This adds the additional implication that profits must be non-negative.

Corollary 6.5 *Given is a production dataset D. Then the following are equivalent:*

I) *For all j,k, $p^k \cdot y^k \geq 0$ and $p^k \cdot y^k \geq p^k \cdot y^j$.*
II) *D is rationalizable by a production set containing* 0.
III) *D is rationalizable by a closed, convex, and comprehensive production set containing* 0.

Proof. Let Y be the convex hull of the elements of D and the origin, and proceed as above.

Theorem 6.4 illustrates an important distinction between the cases of production and demand. We can say that a dataset D satisfies the *weak axiom of production* if for all j,k, $p^k \cdot y^k \geq p^k \cdot y^j$; that is, if condition (I) in Theorem 6.4 is satisfied. The weak axiom of production is also called the *weak axiom of profit maximization*. Note that the weak axiom of production is a binary condition – that is, we only need to check *pairs* of data points to verify its satisfaction. In our analysis of rational demand, we found that WARP was too weak, and that rationalizability instead required GARP. The reason behind the sufficiency of the weak axiom of production lies in the simpler structure of the problem of production. In the case of demand, utility and the "shadow price" on utility are unknowns. The Afriat inequalities of Chapter 3 state that such unknowns must be extracted from the data. In contrast, in the case of production, profit is directly observable from the data.

An interesting and simple result on identification of production sets is possible. Given a rationalizable dataset D, we can define the *lower production set* $\underline{Y}(D)$ to be the set defined in the proof of Theorem 6.4: Let $\underline{Y}(D)$ be the

[1] This is the smallest convex and comprehensive set containing these points. See Figure 6.2(a) for an example.

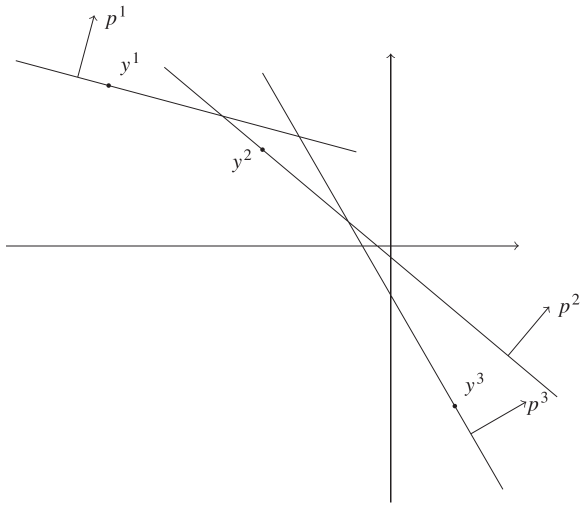
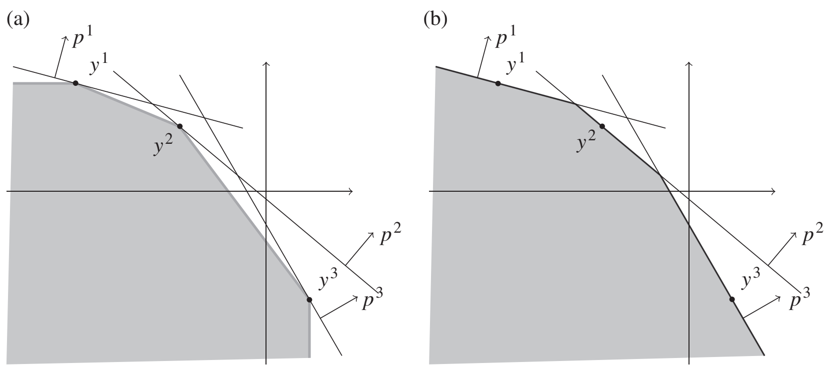


Fig. 6.1 A rationalizable production dataset.



The lower production set, $\underline{Y}(D)$. The upper production set, $Y(\overline{D})$.

Fig. 6.2 Range of rationalizing production sets.

convex and comprehensive hull of the elements y^k of D. We can define the *upper production set* $\overline{Y}(D)$ to be the polyhedron generated by data D; that is, let

$$\overline{Y}(D) = \{y \in \mathbf{R}^n : p^j \cdot y \leq p^j \cdot y^j \text{ for all } j\}.$$

These rationalizing production sets are illustrated in Figure 6.2, based on the example from Figure 6.1.

Theorem 6.6 *Let D be a rationalizable production dataset, and $Y \subseteq \mathbf{R}^n$ be convex and comprehensive. Then Y rationalizes D iff $\underline{Y}(D) \subseteq Y \subseteq \overline{Y}(D)$.*

Proof. It is clear that if Y rationalizes D, then the set inclusions must hold. Conversely, suppose that Y is convex and comprehensive and that the set inclusions hold. Let $y \in Y$. Because $\underline{Y}(D) \subseteq Y$, we know that $y^k \in Y$ for all k. And because $Y \subseteq \overline{Y}(D)$, we know that $p^k \cdot y \leq p^k \cdot y^k$ for all k.

As in the case of demand theory, if one wishes to place additional assumptions on Y, these may add restrictions on the rationalizable datasets. One assumption of interest is constant returns to scale: A production set Y exhibits *constant returns to scale* if for all $y \in Y$ and all $\lambda \in \mathbf{R}_+$, $\lambda y \in Y$. It turns out that the only additional implication imposed by this restriction is that the observed firm always earns zero profits (contrast with the discussion of homotheticity in 4.2.1).

Theorem 6.7 *The following statements are equivalent:*

I) *For all* j,k, $0 = p^k \cdot y^k \geq p^k \cdot y^j$.
II) *D is rationalizable by a production set satisfying constant returns to scale.*
III) *D is rationalizable by a closed, convex, and comprehensive production set satisfying constant returns to scale.*

Proof. That (III) implies (II) is obvious. To see that (II) implies (I), we must have $0 = p^j \cdot y^j$ for all j, otherwise, if $p^j \cdot y^j > 0$, then y^j could not maximize profits. And because a firm can always choose $0 \in Y$ as production, profits must be non-negative.

To see that (I) implies (III), define Y to be the convex, comprehensive cone generated by the vectors y^k of D; that is, the smallest convex and comprehensive cone containing $\{y^k : k = 1, \ldots, K\}$. Note that Y satisfies the required properties. It is clear that $y^k \in Y$ for all k. Now, let $y \in Y$. We will show that $p^j \cdot y^j \geq p^j \cdot y$. There exists $\lambda \in \mathbf{R}^n_+$ for which $y \leq \sum_{k=1}^K \lambda_k y^k$. Now, $p^j \cdot y \leq \sum_{k=1}^K \lambda_k (p^j \cdot y^k) \leq \sum_{k=1}^K \lambda_k (p^j \cdot y^j) = 0 = p^j \cdot y^j$.

Remark 6.8 An interesting distinction from the consumer case is that each of the results of this section could be proved with infinite datasets.

6.2.0.1 Nonlinear pricing

Just as in demand theory, we can consider an environment of nonlinear pricing. To this end, for each k, we suppose we have a *profit function* $g^k : \mathbf{R}^n \to \mathbf{R}$ which is weakly increasing (so that $x \leq y$ implies $g(x) \leq g(y)$) and continuous. In this context, we say that D is rationalizable if there exists Y for which $y^k \in Y$ for all k, and for all $y \in Y$, $g^k(y^k) \geq g^k(y)$.

Theorem 6.9 *For a given production dataset D, the following are equivalent:*

I) *For all* j,k, $g^k(y^k) \geq g^k(y^j)$.
II) *D is rationalizable.*
III) *D is rationalizable by a closed and comprehensive production set.*

Proof. That (III) implies (II) and (II) implies (I) are again obvious. To see that (I) implies (III), define $Y = \{x \in \mathbf{R}^n : g^k(x) \leq g^k(y^k) \text{ for all } k\}$. Note that Y is closed, as the intersection of a collection of closed sets (that is, $Y = \bigcap_{k=1}^{K} \{x \in \mathbf{R}^n : g^k(x) \leq g^k(y^k)\}$, each of which is closed by continuity). Further, it is comprehensive since each g^k is weakly increasing. By (I), for all k, $y^k \in Y$. And if $y \in Y$, then by definition, $g^k(y^k) \geq g^k(y)$.

Theorem 6.9 has an interesting connection with the theory of fairness. If each g^k is a utility function, and y^k is agent k's consumption, then the inequality $g^k(y^k) \geq g^k(y^j)$ states that the allocation $(y^1,...,y^K)$ is *envy-free*. The result characterizes envy-free allocations as those for which there is some set Y from which each agent is allowed to maximize preference.

6.2.1 Unobserved factors of production

In contrast with demand theory (see 3.2.3), we can obtain restrictions from profit maximization even when the choices of some goods are unobserved.

Assume that production takes place in $\mathbf{R}^{m+n}$; prices and output of the first m goods are observed, while of the last n goods, only prices are observed. We shall define a *partial production dataset* to be a collection D of vectors $(y^k,(p^k,\pi^k))$, $k = 1,\dots K$, with $y^k \in \mathbf{R}^m$ and $(p^k,\pi^k) \in \mathbf{R}^{m+n}_{++}$. Say that partial production dataset D is *rationalizable* if there exists $Y \subseteq \mathbf{R}^{m+n}$ and, for each k, $x^k \in \mathbf{R}^n$ such that for all k, $(y^k,x^k) \in Y$ and $(p^k,\pi^k)\cdot(y^k,x^k) \geq (p^k,\pi^k)\cdot(y,x)$ for all $(y,x) \in Y$.

By Theorem 6.4, we know that a partial production dataset is rationalizable iff it is rationalizable by a closed, convex, and comprehensive production set.

Theorem 6.10 *A partial production dataset is rationalizable iff for all* $\Lambda = (\lambda_{j,k}) \in \mathbf{R}^{K\times K}_{+}$ *such that for all* k,

$$\sum_{j\in K} \lambda_{(j,k)}\pi^k = \sum_{j\in K} \lambda_{(k,j)}\pi^j$$

and $\lambda_{(k,k)} = 0$, *we have*

$$\sum_{(j,k)\in K\times K} \lambda_{(j,k)} p^k \cdot (y^j - y^k) \leq 0.$$

Proof. By Theorem 6.4, rationalizability is equivalent to the existence of x^k solving the following inequalities, one for each pair j,k where $j \neq k$:

$$\pi^k \cdot x^k - \pi^k \cdot x^j \geq p^k \cdot (y^j - y^k).$$

By Lemma 1.14, this inequality has no solution iff there are non-negative real numbers $\lambda_{(j,k)}$ such that for all k,

$$\sum_{\{j\in K: j\neq k\}} \lambda_{(j,k)}\pi^k - \sum_{\{j\in K: j\neq k\}} \lambda_{(k,j)}\pi^j = 0$$

and

$$\sum_{\{(j,k)\in K\times K: j\neq k\}} \lambda_{(j,k)} p^k \cdot (y^j - y^k) > 0.$$

This is exactly what the statement of the theorem precludes (note that $\lambda_{(k,k)}$ is unconstrained so we can take $\lambda_{(k,k)} = 0$).

Theorem 6.10 has two immediate important corollaries. The first is the following negative result, which concerns prices of unobserved commodities that are *conically independent* across observations, in the sense that no such price vector is in the convex cone spanned by the remaining price vectors. For example, every linearly independent set of vectors is conically independent, but not conversely. In the case of conic independence, there are no testable implications to the profit maximization hypothesis. This is problematic if we do not know which factors may not be observed. As a practical matter, it seems likely that the larger the number of unobserved commodities, the more likely the price vectors are to be conically independent. In such an environment, the hypothesis of profit maximization is not falsifiable.

Corollary 6.11 *Suppose that the vectors $\{\pi^k\}_{k=1}^K$ have the property that for each j, π^j is not in the convex cone spanned by $\{\pi^k\}_{k\neq j}$. Then $(y^k,(p^k,\pi^k))$ is rationalizable.*

The second is the case in which for all j,k, we have $\pi^j = \pi^k$. The matrix Λ in Theorem 6.10 then satisfies the condition that, for all k, $\sum_{\{j:j\neq k\}} \lambda_{(j,k)} = \sum_{\{j:j\neq k\}} \lambda_{(k,j)}$. The diagonal of Λ is identically zero. Consider a modified matrix Λ' which differs only from Λ on the diagonals, and the diagonals are chosen to be non-negative and so that for all j,k, $\sum_{\{i\in K\}} \lambda_{(i,j)} = \sum_{\{i\in K\}} \lambda_{(i,k)}$. The matrix Λ' now has the feature that all columns and rows sum to the same number. In combinatorics, such a matrix is a non-negative multiple of a *bistochastic matrix*.[2] The reason this is of interest is that, by a theorem of Birkhoff and von Neumann (which can be found in Berge 1963, for example), a bistochastic matrix is known to be a convex combination of *permutation* matrices.[3]

It therefore follows that Λ' is a non-negative linear combination of permutation matrices. As a result of this, it is easily seen that the condition in Theorem 6.10 reverts to a cyclic monotonicity condition as in Theorem 1.9. However, the inequality here is of the opposite sign as the version in Theorem 1.9. This should not be a surprise, as the following corollary relates to profit maximization, and the profit function is well known to be convex, rather than concave.

[2] A matrix is bistochastic if it consists of only non-negative entries, and the row and column sums are equal to one.

[3] A permutation matrix is a matrix of ones and zeroes, with exactly one 1 in each row and each column.

Corollary 6.12 *Suppose that for all j,k, $\pi^j = \pi^k$. Then D is rationalizable iff for all sequences $i_1, ..., i_k$,*

$$\sum_{j=1}^{k} p^{i_j} \cdot (y^{i_{j+1}} - y^{i_j}) \leq 0.$$

It is possible to give a "cyclic" version (one ruling out certain cycles, that is) of Theorem 6.10 in the general case, but this condition does not seem to be any more illuminating than the one stated in the theorem.

Similar exercises can be undertaken with partial production datasets, which can derive the testable implications of constant returns to scale and other such hypotheses. We shall not undertake such an exercise here.

6.2.2 Measuring violations from rationalizability

The most common approach to measuring deviations from rationalizability is to find an approximate technology which would have generated the observed data. There are generally two approaches: a most conservative, and a least conservative, corresponding to the ideas in Theorem 6.6. Given an approximate production set, a measure of efficiency as in Debreu (1951) or Farrell (1957) can be applied to the observed data given this production set.

Banker and Maindiratta (1988) suggest that one should construct a technology so that as many data points as possible are consistent with profit maximization. So, they ask that all observed vectors are feasible for the technology, and as many of possible maximize profits. They establish that $\underline{Y}(D)$ generates such a technology. The axiomatic approach of Banker, Charnes, and Cooper (1984) justifies $\underline{Y}(D)$.[4]

Varian (1990) describes a very similar idea. That is, if there is a pair (p^k, y^k) (p^j, y^j) for which $p^k \cdot y^k < p^k \cdot y^j$, Varian recommends that the measure of inefficiency for this violation should be $\frac{p^k \cdot y^j}{p^k \cdot y^k} - 1$. This measure is economically meaningful, in that it represents the percentage gain in profit that the firm could have obtained by producing y^j instead of y^k.

6.3 CHAPTER REFERENCES

Theorem 6.1 can be found in Hanoch and Rothschild (1972) and Varian (1984). Hanoch and Rothschild (1972) considers the case when the rationalizing production function is only required to be weakly monotonic.

Theorems 6.4 and 6.7 and Corollary 6.5 are also found in Hanoch and Rothschild (1972). Again, Varian (1984) also discusses this result.

[4] Their axioms are convexity, comprehensivity, $y^k \in Y$ for all k, and minimality with respect to set inclusion while satisfying these properties. That is, their axioms define the convex, comprehensive hull of y^k.

Theorems 6.2 and 6.3 are basically due to Afriat (1972) and Diewert and Parkan (1983). The idea behind Theorem 6.6 is due to Diewert and Parkan (1983) and Varian (1984).

The ideas of this section first appear in Afriat (1972) and Hanoch and Rothschild (1972). These papers also consider the case where no price data are observed; and the goal is simply to test whether there is a production function or production set consistent with the observed inputs and outputs being produced efficiently. Tests where there are data only on certain of the variables are considered in detail in Afriat (1972), Hanoch and Rothschild (1972), and Diewert and Parkan (1983). Theorem 6.10 is based on the linear programming ideas found in these works. Issues we have not discussed, which are discussed in detail in these papers, are the implications of assumptions on production when profits may be observable, or when no prices are observable, and so on. Färe, Grosskopf, and Lovell (1987), for example, talks about tests for non-monotonicity.

Many of the early results described in this chapter have evolved into the field of *data envelopment analysis* (DEA), which seeks to measure and test efficiency of productive units via linear programming and convex analytic techniques. See, for example, Cooper, Seiford, and Tone (2007). *Stochastic frontier analysis* studies similar questions from a stochastic production standpoint; see Kumbhakar and Lovell (2003) for an exposition.

The ideas on which production set to use for efficiency measurement are due to Banker, Charnes, and Cooper (1984) and Banker and Maindiratta (1988). These ideas are fundamental in DEA. The measure of inefficiency of observed data briefly discussed is due to Varian (1990).

One may be interested in studying particular properties of the unobserved technologies: The papers by Chambers and Echenique (2009a) and Dziewulski and Quah (2014) focus on supermodularity of the production function.

There is a large empirical literature testing the ideas in this chapter in a nonparametric fashion. A few such works which test the ideas in agricultural settings include Fawson and Shumway (1988), Färe, Grosskopf, and Lee (1990), Ray and Bhadra (1993), Featherstone, Moghnieh, and Goodwin (1995), and Tauer (1995).

The Birkhoff–von Neumann Theorem is an important result in graph theory, and is attributed to Birkhoff (1946) and Von Neumann (1953).