CHAPTER 9

General Equilibrium Theory

The previous chapters deal with theories of individual agents' behavior. In the rest of the book, we turn to economic theories that predict group or societal outcomes. We first turn our attention to general equilibrium theory.

General equilibrium theory can often be studied through a reduced-form model, the excess demand function of an economy. The equilibrium outcomes of the economy are given as zeroes of the excess demand function. There are two immediate questions about the scope of the model: What is the class of excess demand functions that can arise from a well-behaved economy? And which sets of prices can be equilibrium prices?

The answers to these questions carry a largely negative message about general equilibrium theory. The Sonnenschein–Mantel–Debreu Theorem (as we shall refer to it) shows that, roughly speaking, any continuous function that satisfies Walras' law can be the aggregate excess demand function of a very well-behaved economy. The result implies that any compact set of strictly positive prices can be the set of Walrasian equilibrium prices of a well-behaved economy. No additional constraints are obtained by insisting on basic regularity properties of the equilibria.

Considered as data on an economy, an excess demand function, or a set of putative equilibrium prices, may seem odd. The next set of questions under study is much more similar to the approach in Chapter 3. If we assume that we can observe equilibria for different vectors of endowments (in a sense, we can sample from the "equilibrium manifold"), then the theory of general equilibrium can be refuted: There are nonrationalizable datasets. The theory is testable if we can observe prices from different endowment vectors. The nature of the testable implications follow from a very general principle, the Tarski–Seidenberg Theorem, which we shall also review here.

We focus on a model of an economy where all economic activity takes the form of exchange. There are I consumers; each consumer i is described by a pair $(\succeq_i, \omega_i)$, where $\succeq_i$ is a preference relation on $\mathbf{R}^n_+$, and $\omega_i \in \mathbf{R}^n_+$ is an *endowment vector*. An *exchange economy* is a tuple $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$.

When $\succeq_i$ is continuous, strictly convex, and locally nonsatiated, the *demand function* of agent i is well defined as $d^i(p,M) = \arg\max_{\succeq_i}\{x \in \mathbf{R}^n_+ : p \cdot x \leq M\}$, for $p \in \mathbf{R}^n_{++}$ and $M > 0$. The *excess demand function* of agent i is $Z^i(p) = d^i(p, p \cdot \omega_i) - \omega_i$. Finally, the *aggregate excess demand function* of the economy $\mathcal{E}$ is $Z = \sum_{i=1}^I Z^i$. The function Z is continuous and satisfies $p \cdot Z(p) = 0$ for all p in its domain, a property called *Walras' Law*. A vector $p \in \mathbf{R}^n_{++}$ is a *Walrasian equilibrium price* if $Z(p) = 0$.

One can define the weak axiom of revealed preference for excess demand functions. Indeed, note that WARP for individual demand functions says that there cannot exist prices p and q such that $q \cdot d^i(p, p \cdot \omega_i) < q \cdot \omega_i$ and $p \cdot d^i(q, q \cdot \omega_i) \leq p \cdot \omega_i$. So we say that an individual excess demand function Z^i satisfies WARP if there are no p and q such that $q \cdot Z^i(p) < 0$ and $p \cdot Z^i(q) \leq 0$. Moreover, Z^i satisfies a version of the strong axiom of revealed preference, which states that there is no sequence of prices $p_1, \ldots p_K$ with $p_k \cdot Z^i(p_{k+1}) \leq 0$, $k = 1, \ldots,$ $K-1$ and $p_K \cdot Z^i(p_1) < 0$.

9.1 THE SONNENSCHEIN–MANTEL–DEBREU THEOREM

We know that the aggregate excess demand function of an exchange economy satisfies Walras' Law and has homogeneity of degree zero. We ask here whether there are any other properties which are systematically satisfied by excess demand. Let $S = \{p \in \mathbf{R}^n_+ : \|p\| = 1\}$ be the intersection of the unit sphere with the non-negative orthant, and let $\Delta = \{p \in \mathbf{R}^n_+ : \sum_i p_i = 1\}$. By homogeneity, one can, without loss of generality, restrict attention to prices either in S or Δ, as demand functions and excess demand functions are homogeneous of degree zero. Depending on the context, it is easier to work with S or Δ. The relative interior of S is denoted by int S. Likewise the relative interior of Δ is denoted by int Δ.

Sonnenschein–Mantel–Debreu Theorem *Suppose $Z : S \to X$ is a continuous function satisfying Walras' Law and let $K \subseteq \text{int } S$ be compact. Then there exists an exchange economy $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^n$, in which each $\succeq_i$ is continuous, strictly convex, and monotonic, such that the sum of individuals agents' excess demand functions in $\mathcal{E}$ coincides with Z on K.*

The Sonnenschein–Mantel–Debreu (SMD) Theorem has a complicated proof, but we present the gist of it in Section 9.1.1. Note that the economy $\mathcal{E}$ in the SMD Theorem has a number of agents that is equal to the number of goods. The next result shows that the conclusion of the theorem does not hold with fewer consumers.

Proposition 9.1 *There is a function Z satisfying the hypotheses of the SMD Theorem, and a nonempty compact set $K \subseteq \text{int } S$ such that Z cannot be written as the sum of fewer than n individual agents' excess demand functions on K.*

Proof. For any $p \in S$, let $T(p)$ denote the subspace that is orthogonal to p in $\mathbf{R}^n_+$. Choose $p^0 \in \operatorname{int} S$ arbitrarily, and let $\varepsilon > 0$ be such that $p^0_i > \varepsilon$ for all i. Let K be the set of $p \in \operatorname{int} S$ with $p_i \geq \varepsilon$ for all i. The set K is compact.

Define the function $Z : S \to \mathbf{R}^n_+$ by letting $Z(p)$ be the projection of the vector $(p - p^0)$ on to $T(p)$.[1] Note that Z is continuous and satisfies Walras' Law, as $p \cdot Z(p) = 0$ because $Z(p)$ is orthogonal to p. Observe that, for any $p \in S \backslash \{p^0\}$,

$$(p - p^0) \cdot Z(p) > 0, \tag{9.1}$$

because $p - p^0$ can never be orthogonal to p, as both p and p^0 are in the unit sphere. This is a basic property of projections.

Suppose, toward a contradiction, that there are $k < n$ individual excess demand functions such that $Z(p) = \sum_{i=1}^k Z^i(p)$ for all $p \in K$. Walras' Law implies that $Z^i(p^0) \in T(p^0)$. The vectors $Z^i(p^0)$, $i = 1, \ldots, k$, form a linearly dependent set, as $0 = Z(p^0) = \sum_{i=1}^k Z^i(p^0)$. Let Λ be the linear subspace of $T(p^0)$ spanned by the vectors $Z^i(p^0)$, $i = 1, \ldots, k$.

The dimension of Λ is strictly smaller than the dimension of $T(p^0)$, because $\dim \Lambda < k \leq n - 1 = \dim T(p^0)$. Then the orthogonal complement $\Lambda^\perp$ of Λ in $T(p^0)$ is nontrivial. Since p^0 projects to 0 in $T(p^0)$, one can choose $\eta \in \Lambda^\perp$ small enough so that there is $\bar{p} \in K$ such that $\bar{p} \neq p^0$, and $\bar{p} - p^0$ projects to η in $T(p^0)$.

Choose an arbitrary i. Then

$$(\bar{p} - p^0) \cdot Z^i(p^0) = (\bar{p} - p^0 - \eta) \cdot Z^i(p^0) + \eta \cdot Z^i(p^0) = 0.$$

This follows because we know $(\bar{p} - p^0 - \eta) \cdot Z^i(p^0) = 0$ as $Z^i(p^0) \in T(p^0)$ and $(\bar{p} - p^0 - \eta)$ is the orthogonal projection of $(\bar{p} - p^0)$ onto p^0. Further, we know $\eta \cdot Z^i(p^0) = 0$ as $Z^i(p^0) \in \Lambda$ and $\eta \in \Lambda^\perp$. The function Z^i satisfies the weak axiom of revealed preference, because it is an individual agent's excess demand function. Then by Walras' Law, $(\bar{p} - p^0) \cdot Z^i(p^0) = 0$ implies that $\bar{p} \cdot Z^i(p^0) \leq 0$. By the weak axiom then, $p^0 \cdot Z^i(\bar{p}) \geq 0 = \bar{p} \cdot Z^i(\bar{p})$. Thus $(\bar{p} - p^0) \cdot Z^i(\bar{p}) \leq 0$. Since this holds for all i, we obtain that $(\bar{p} - p^0) \cdot Z(\bar{p}) \leq 0$, in contradiction of (9.1).

The SMD Theorem talks about the behavior of Z on a compact subset of the sphere; but students of general equilibrium theory know that many results rely on the behavior of Z close to the boundary of its domain, when some prices are close to zero. The next theorem says that one can decompose Z on all of its domain as the sum of individual excess demand functions. The decomposition is of a weaker nature, though.

Theorem 9.2 *Suppose $Z : S \to X$ is a function satisfying Walras' Law that is bounded below. Then there are functions $Z^i : S \to X$, $i = 1, \ldots, n$, satisfying Walras' Law and the Strong Axiom of Revealed Preference, for which $Z = \sum_{i=1}^n Z^i$.*

[1] Formally, $Z(p) = p - p^0 - pp'(p - p^0) = pp'p^0 - p^0$, since $||p|| = 1$; p' is the transpose of p.

A proof of Theorem 9.2 can be obtained with arguments similar to those in Section 9.1.1.

The SMD Theorem says that general equilibrium theory does not restrict aggregate excess demand functions. It may not seem natural to assume that someone could "observe" an excess demand function. It makes more sense that one could observe a set of putative equilibrium prices. The theorem implies, however, that any compact set of prices can be Walrasian equilibrium prices:

Corollary 9.3 *Let $K \subseteq \text{int } S$ be compact. For any compact set $K' \subseteq \text{int } S$ with $K \subseteq K'$, there exists an exchange economy $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^n$, in which each $\succeq_i$ is continuous, strictly convex, and monotonic, for which K is the set of Walrasian equilibrium prices of $\mathcal{E}$ in K'.*

Proof. Let $g(p) = (\mathbf{1}_1 - pp'\mathbf{1}_1)$ (one of the projections in the proof of the SMD Theorem in Section 9.1.1, below),

$$f(p) = \inf_{q \in K} \|p - q\|,$$

and set $Z(p) = f(p)g(p)$. Note that Z is continuous, $p \cdot Z(p) = f(p)(p \cdot g(p)) = 0$, and $Z(p) = 0$ iff $p \in K \cup \{\mathbf{1}_1\}$. So for $p \in \text{int } S$, $Z(p) = 0$ iff $p \in K$. The result follows from the SMD Theorem applied to Z and K'.

Some classic results of Mas-Colell strengthen Corollary 9.3. First, he shows that any compact set in the interior of the price sphere can be the equilibrium price set of a well-behaved economy (even taking the boundary into consideration). Further, if there are at least three commodities, then for any finite set $A \subseteq \text{int } S$ with an odd number of elements, and any function $d : A \to \{-1, 1\}$ such that $\sum_{p \in A} d(p) = 1$, there is a well-behaved exchange economy with a smooth excess demand function such that A is its set of Walrasian equilibrium prices, and $d(p)$ is the index (see Mas-Colell, Whinston, and Green, 1995) of equilibrium p.

9.1.1 Sketch of the proof of the Sonnenschein–Mantel–Debreu Theorem

Consider first the excess demand function of a single consumer, $Z(p) = d(p, p \cdot \omega) - \omega$. As discussed at the beginning of the chapter, the excess demand function Z satisfies WARP if there are no p and q such that $q \cdot Z(p) < 0$ and $p \cdot Z(q) \leq 0$. Observe that if Z satisfies the weak axiom of revealed preference, and $g : S \to \mathbf{R}_+$ is a strictly positive function, then $p \mapsto g(p)Z(p)$ also satisfies the weak axiom of revealed preference.

In preparation for the proof, we need to consider the function $\tilde{Z}^i(p) = \mathbf{1}_i - p_i p$. We intend for $\tilde{Z}^i$ to be the excess demand function of an individual consumer. Recall that $\mathbf{1}_i$ is the unit vector with a one in the ith coordinate. Recall that $\|p\| = 1$ for $p \in S$; thus for any vector $x \in X$, $pp'x$ is the orthogonal projection of x onto the subspace defined by the vector p. To see this, note that

pp' satisfies $(pp')(pp') = pp'$, so that for any x, if $y = pp'x$, then $y = pp'y$ (a property called idempotence). Moreover, for any x, $pp'(x - pp'x) = 0$. So

$$\tilde{Z}^i(p) = \mathbf{1}_i - pp'\mathbf{1}_i = \mathbf{1}_i - p_i p$$

is the "residual" of projecting $\mathbf{1}_i$ on the linear subspace spanned by p. In other words, it is the projection of $\mathbf{1}_i$ on the subspace of vectors orthogonal to p. Observe that $\tilde{Z}^i$ is continuous and satisfies Walras' Law, as $p \cdot \tilde{Z}^i(p) = p_i - p_i p \cdot p = 0$.

Note that $\tilde{Z}^i$ is obtained as the result of a maximization program because $\tilde{Z}^i(p)$ is the projection of $\mathbf{1}_i$ on to a subspace. (The projection results from maximizing the negative of the distance of $\mathbf{1}_i$ to the subspace of vectors orthogonal to p.) As a consequence of being the solution to a maximization program, $\tilde{Z}^i$ will satisfy the weak axiom of revealed preference.

So we see that $\tilde{Z}^i$ is continuous, and satisfies Walras' law and WARP. The idea in the rest of the proof is to use the functions $\tilde{Z}^i$ as a "basis" on which one can decompose the function Z.

The set K from the hypothesis of the theorem is a compact set in the interior of the sphere. Let $f : S \to \mathbf{R}$ be a continuous function for which

$$Z_i(p) + f(p)p_i > 0$$

for all $p \in K$.[2] There is such a function because K is compact and $p_i > 0$. Note that the function

$$p \mapsto (Z_i(p) + f(p)p_i)\tilde{Z}^i(p) = (Z_i(p) + f(p)p_i)(\mathbf{1}_i - p_i p)$$

satisfies the weak axiom of revealed preference on K by the observation made above. Define then $Z^i(p) = (Z_i(p) + f(p)p_i)(\mathbf{1}_i - p_i p)$.

We are not going to prove that $Z^i(p)$ on K can be generated by preferences $\succeq_i$ which are continuous, strictly convex, and monotonic. This proof is quite involved and depends on the fact that K is a compact subset of the relative interior of the unit sphere. One can see that it is "almost" generated by preferences satisfying these properties (since it is based on a maximization problem). We hope that having established that $Z^i(p)$ satisfies the weak axiom is instructive enough.

Finally, we must verify that $Z = \sum_{i=1}^n Z^i$. Observe that

$$\begin{aligned}
&\sum_{i=1}^n (Z_i(p) + f(p)p_i)(p_i p) \\
&= \sum_{i=1}^n (p_i Z_i(p) + p_i^2 f(p))p \\
&= (p \cdot Z(p) + p \cdot p f(p))p = f(p)p.
\end{aligned}$$

[2] Note that Z_i is the ith component of the function Z, not to be confused with Z^i.

Likewise,

$$\sum_{i=1}^{n} (Z_i(p) + f(p)p_i)\,\mathbf{1}_i$$
$$= Z(p) + f(p)p.$$

Thus,

$$\sum_{i=1}^{n} (Z_i(p) + f(p)p_i)\,(\mathbf{1}_i - p_i p) = Z(p).$$

9.2 HOMOTHETIC PREFERENCES

We now turn to a different (and simpler) construction than the one in SMD. It requires that Z be smooth, but it delivers a rationalizing economy in which all agents' preferences are homothetic.

Homotheticity has, generally speaking, strong implications. For example, it is a crucial ingredient in aggregation theorems, from which the existence of a representative consumer follows. It is therefore striking that there is a version of the SMD Theorem even when we ask that agents' preferences be homothetic.

For convenience, we now take the domain of Z to be Δ. The next result is due to Rolf Mantel.

Theorem 9.4 *Suppose that $Z : \Delta \to X$ is a C^2 function satisfying Walras' Law, and let $K \subseteq \text{int}\, \Delta$ be compact. Then there exists an exchange economy $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^n$, in which each $\succeq_i$ is continuous, convex, homothetic, and monotonic, such that the sum of individuals agents' excess demand functions in $\mathcal{E}$ coincides with Z on K.*

Proof. Let Z be as in the hypothesis of the theorem. We construct an exchange economy with n agents, each one endowed with m units of one good: the endowment of agent i is $m\mathbf{1}_i$. As we shall see, the parameter m plays an important role in this construction because it scales up an economy in which all prices are equilibrium prices.

Let A be an $n \times n$ matrix and denote by a_i the vector formed from its ith column. We choose A such that the a_i vectors are linearly independent and $a_i \cdot \mathbf{1} = 1$ (we use the notation $\mathbf{1}$ for a vector of ones). For an n-vector x, $\log(x)$ denotes the vector whose entries are the logarithms of the entries of x.

Define the following functions:

$$g^i(p) = \frac{1}{m} Z_i\left(\frac{1}{\sum_j p_j} p\right) - a_i \cdot \log(Ap),$$

on the domain consisting of $p \in \mathbf{R}^n_{++}$ with $\frac{1}{\sum_j p_j} p \in K$. Denote by K' this domain of prices. We intend g^i to be the indirect utility function of agent i in our construction, when her income is 1.

Observe now that $a_i \log(Ap)$ is a concave function of p, and that the second derivatives of Z are bounded on K. Then by choosing m large enough we know that $g^i(p)$ is a convex function on K.

Define the utility function of agent i to be

$$u^i(x) = \inf\{g^i(p) : p \cdot x \le 1, p \in K'\}.$$

It is routine to verify that u^i is monotonic, continuous, and quasiconcave. To see that the preferences represented by u^i are homothetic, note that u^i is log-homogeneous:

$$\begin{aligned} u^i(\lambda x) &= \inf\{g^i(p) : \lambda p \cdot x \le 1, p \in K'\} \\ &= \inf\{g^i((1/\lambda)q) : q \cdot x \le 1, q \in K'\} \\ &= \inf\{\frac{1}{m} Z_i\left(\frac{1}{\sum_j q_j} q\right) - a_i \cdot \log(Aq(1/\lambda)) : q \cdot x \le 1, q \in K'\}, \end{aligned}$$

but

$$a_i \cdot \log(Aq(1/\lambda)) = a_i \cdot \log(Aq) - a_i \cdot \mathbf{1} \log(\lambda) = a_i \cdot \log(Aq) - \log(\lambda),$$

so that $u^i(\lambda x) = u^i(x) + \log(\lambda)$. This means that a monotonic transformation of u^i is homogeneous, and therefore that the preferences represented by u^i are homothetic.

As a result, if we let $v^i(p,M)$ be the indirect utility function derived from u^i, then

$$v^i(p,M) = g^i(p) + \log(M),$$

for $M > 0$.

Using Roy's identity we obtain the demand of agent i as

$$d_i(p, p \cdot m\mathbf{1}_i) = -\frac{\nabla_p v^i(p, p \cdot m\mathbf{1}_i)}{\nabla_M v^i(p, p \cdot m\mathbf{1}_i)} = -p \cdot m\mathbf{1}_i \nabla_p g^i(p).$$

Now,

$$\nabla_p g^i(p) = \frac{1}{m} \nabla Z_i(p) - A' L(p)^{-1} a_i,$$

where $L(p)$ is the $n \times n$ diagonal matrix which has $\sum_{j=1}^n a_{ji} p_j$ in its ith row and column. So

$$d_i(p, p \cdot m\mathbf{1}_i) = -p \cdot m\mathbf{1}_i \nabla g^i(p) = -p_i \nabla Z_i(p) + (p_i m) A' L(p)^{-1} a_i.$$

Then aggregate excess demand for this economy is

$$\begin{aligned} \sum_{i=1}^n d_i(p, p \cdot m\mathbf{1}_i) - m\mathbf{1} &= -\sum_{i=1}^n p_i \nabla Z_i(p) + \sum_{i=1}^n (p_i m) A' L(p)^{-1} a_i - m\mathbf{1} \\ &= Z(p) + mA' L(p)^{-1} Ap - m\mathbf{1} \\ &= Z(p) + mA'\mathbf{1} - m\mathbf{1} = Z(p), \end{aligned}$$

where we have used the facts that $Z(p) + \sum_{i=1}^n p_i \nabla Z_i(p) = 0$ (which follows from Walras' Law) and that $A'\mathbf{1} = \mathbf{1}$.

Remark 9.5 The proof of Theorem 9.4 uses an interesting construction. It uses an economy in which agents' preferences are homothetic and all prices are equilibrium prices. One can verify that if g^i in the proof is defined to be $a_i \cdot \log(Ap)$, and agents' endowments are as above, then the construction used in the proof gives an economy in which all prices are zero.

The proof works by adding a scaled-down version of $Z(p)$ to the excess demand function, and rationalizes such a "perturbed" excess demand by a perturbation of the original homothetic preferences. The new economy has the zeroes of Z as equilibrium prices, but its excess demand function is a scaled-down version of Z. Now homotheticity guarantees that by scaling up endowments we obtain an economy in which Z is the excess demand function.

9.3 PRICES AND ENDOWMENTS

We have looked at the implications of general equilibrium theory when one is given either a set of prices or an aggregate excess demand function. We now turn to a different set of givens.

We assume that we observe a finite collection of prices and endowments (or, equivalently, of aggregate endowment and individual agents' incomes). Under the assumption that one can observe prices and endowments, we are going to show that general equilibrium theory is testable. This observation is due to Brown and Matzkin, as is most of the discussion in Section 9.3.

Consider an exchange economy $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$, where each pair $(\succeq_i, \omega_i)$ describes one consumer; as before, each agent i is described by a preference relation $\succeq_i$ and a vector of endowments $\omega_i \in \mathbf{R}^n_+$. I is a positive integer specifying the number of consumers. An *allocation of* $\sum_i \omega_i$ is a vector $(x_i)_{i \in I} \in \mathbf{R}^{nI}_+$ such that $\sum_i x_i = \sum_i \omega_i$. A *Walrasian equilibrium* is a pair $((x_i), p)$ such that (x_i) is an allocation and $p \in \mathbf{R}^l_{++}$ satisfies that x_i is maximal for $\succeq_i$ in the set

$$\{z \in \mathbf{R}^n_+ : p \cdot z \leq p \cdot \omega_i\}.$$

We assume that we have data on prices, incomes, and resources. Specifically, an *economy-wide dataset* is a collection $D_W = (p^k, (\omega_i^k)_{i=1}^I)_{k=0}^K$. If it seems unreasonable to assume that individual endowments are observable, note that one can instead work with individual incomes and aggregate endowment. The results will be the same.

A dataset D_W is *Walras rationalizable* if there are locally nonsatiated preference relations $\succeq_i$ and, for each k an allocation (x_i^k) of $\sum_i \omega_i^k$ such that $((x_i^k), p^k)$ is a Walrasian equilibrium of the exchange economy $(\succeq_i, \omega_i^k)_{i=1}^I$.

There are economy-wide datasets that are not Walras rationalizable. Consider for example the dataset represented in Figure 9.1. In the figure, there are two observations $(p^k, (\omega_1^k, \omega_2^k))$, $k = 1, 2$. For each k, $\bar{\omega}^k = \omega_1^k + \omega_2^k$ defines an Edgeworth box. Suppose that $k = 1$ gives the taller box, while $k = 2$ defines the wider of the two boxes. The boxes are represented so that the consumption space of agent 1 is the same in the two boxes (the $(0,0)$ consumption vector

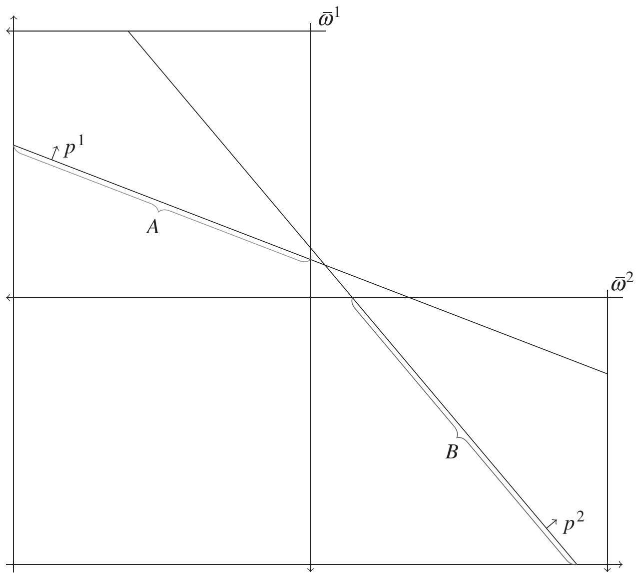

Fig. 9.1 Two Edgeworth boxes representing a non-rationalizable dataset.

for agent 1 is the same in the two boxes). Note that p^k and ω_i^k define a budget set for each consumer.

If the data in Figure 9.1 were Walras rationalizable, there would need to exist some (unobserved) allocation of $\bar{\omega}^k$, for $k = 1,2$. These allocations must lie inside each Edgeworth box. Then, if the dataset in Figure 9.1 were Walras rationalizable, the allocation of $\bar{\omega}^1$ would have to lie on the segment A of the budget line for consumer 1, while the allocation of ω^2 would have to lie on the segment B of the budget line for consumer 1 at prices p^2. Any such allocation would imply that consumer 1 violates WARP. So the allocation could not be part of a Walrasian equilibrium, as the resulting choices by consumer 1 would be incompatible with any preference relation for consumer 1.

The observation that the theory of Walrasian equilibrium is testable is subtle, and in sharp contrast with the message of the SMD Theorem, but it is not different in nature than the idea that there are individual observations that violate WARP. One would ideally want to characterize all datasets that are Walras rationalizable: such a test exists, but a "closed form" test is not known.

The following theorem (due to Brown and Matzkin) is a direct consequence of Afriat's Theorem. It is useful because it sets up the problem of Walras-rationalizing data as a system of polynomial inequalities.

Theorem 9.6 *A dataset $D_W = (p^k, (\omega_i^k)_{i=1}^I)_{k=0}^K$ is rationalizable iff there are numbers $(U_i^k, \lambda_i^k)_{i=1}^I$ for $k = 1,\ldots,K$, and vectors $(x_i^k)_{i=1}^I \in X^I$ for $k = 1,\ldots,K$,* *such that*

$$\begin{aligned} U_i^k &\le U_i^l + \lambda_i^l p^l \cdot (x_i^k - x_i^l) \\ \lambda_i^k &> 0 \\ p^k \cdot x_i^k &= p^k \cdot \omega_i^k \\ \sum_i x_i^k &= \sum_i \omega_i^k. \end{aligned}$$

The characterization in Theorem 9.6 is not practical as a test because to determine if a dataset is rationalizable requires solving a large system of polynomial inequalities. This is hard to do. The point made by Brown and Matzkin is that there exists a different test: one that is similar in nature to checking GARP or SARP. We proceed to introduce the mathematical framework in which we can express such a test. We need to introduce the mathematical theory that deals with systems of polynomial inequalities.

Consider a system of *polynomial inequalities:*

$$I : \begin{cases} p_1(a,x)\; S_1\; 0 \\ \quad\vdots \\ p_k(a,x)\; S_k\; 0, \end{cases}$$

where $a \in \mathbf{R}^m$ and $x \in \mathbf{R}^n$, so that p_i is a polynomial of $m+n$ variables and $S_i \in \{\geq, =, >, \neq\}$. Here we interpret $a \in \mathbf{R}^m$ as a parameter, and the variables $x \in \mathbf{R}^n$ as unknowns: We want to know whether, for given a, there is x such that (a,x) solves system I.

The following is a celebrated theorem due to Tarski and Seidenberg.

Theorem 9.7 *There are systems of polynomial inequalities $J_1, \ldots, J_L$ in m variables with the following property: For all $a \in \mathbf{R}^m$, there is $x \in \mathbf{R}^n$ for which (a,x) solves system I iff there is l such that a solves system J_l.*

Remark 9.8 The theorem actually says more. It says that if we fix a real closed field F, then there is a solution to I in F iff the coefficients of I are a solution to some J_l in F. Perhaps more importantly, the theorem is not just an existence result. It is also associated with an algorithm which can be used, in principle, to derive the systems $J_1, \ldots, J_L$ from the system I. The algorithm is, however, not computationally efficient. It is known that the problem of eliminating quantifiers for real closed fields in general is computationally complex. We discuss related ideas in Chapter 12.

We present two examples to illustrate the theorem. The first is an example in elementary algebra. Consider the second-degree polynomial of one variable: $p(x) = ax^2 + bx + c$. Suppose that $a \neq 0$. The equation $p(x) = 0$ has a solution (in $\mathbf{R}$) iff the inequality $b^2 - 4ac \geq 0$ is satisfied. The example is an instance of *quantifier elimination* since the statement $\exists x(p(x) = 0)$, which involves an existential quantifier over x, is seen to be equivalent (when $a \neq 0$)

to the statement $b^2 - 4ac \geq 0$.[3] We can call the statement $\exists x(p(x) = 0)$ an *existential* statement because it is preceded by an existential quantifier; while $b^2 - 4ac \geq 0$ is non-existential. As a statement that describes the empirical content of a theory, $\exists x(p(x) = 0)$, or any other existential statement, is very problematic because its verification requires checking all possible real numbers. By eliminating quantifiers we can go from an existential statement to a statement that can be directly verified on the data. Chapter 13 discusses this issue in more depth.

The second example is familiar from our discussion of demand theory in Chapter 3. Let $D = \{(x^k, p^k)\}_{k=1}^K$ be a consumption dataset. Afriat's Theorem says that there is a solution $(U^1, \ldots, U^K, \lambda^1, \ldots, \lambda^K)$ to the (linear) system of Afriat inequalities

$$U^k \leq U^l + \lambda^l p^l \cdot (x^k - x^l), k \neq l$$

and $\lambda^k > 0$ iff the data satisfy GARP. Afriat's Theorem is an example of elimination of quantifiers: The statement that there is a solution to the system of Afriat inequalities is existential. And Afriat's Theorem says that such an existential system is equivalent to GARP – observe that GARP is the statement that for any sequence $x^{h_1}, \ldots, x^{h_H}$ of distinct observations $p^{h_1} \cdot x^{h_1} \geq p^{h_1} \cdot x^{h_2}, \ldots, p^{h_{H-1}} \cdot x^{h_{H-1}} \geq p^{h_{H-1}} \cdot x^{h_H}$ implies that $p^{h_H} \cdot x^{h_H} \leq p^{h_H} \cdot x^{h_1}$. Saying that a dataset satisfies GARP is not an existential statement.

We can formulate the satisfaction of GARP as the satisfaction of a system of polynomial inequalities. To write down this system, let s denote a sequence $x^{h^s_1}, \ldots, x^{h^s_{H^s}}$ of distinct observations in D. Let Σ be the set of all such sequences. For each $s \in \Sigma$ we need to check that there is not a cycle $x^{h^s_1} \succeq^R \ldots \succeq^R x^{h^s_{H^s}} \succ^R x^{h^s_1}$. Satisfaction of GARP is the same as saying that no sequence $s \in \Sigma$ gives rise to a cycle.

Let r^i_s be the inequality $p^{h^s_i} \cdot x^{h^s_i} < p^{h^s_i} \cdot x^{h^s_{i+1}}$, $i = 1, \ldots H^s - 1$, and q_s the inequality $p^{h^s_{H^s}} \cdot x^{h^s_{H^s}} \leq p^{h^s_{H^s}} \cdot x^{h^s_1}$. Note that, for a sequence s, the satisfaction of any one of these inequalities rules out the potential cycle $x^{h^s_1} \succeq^R \ldots \succeq^R x^{h^s_{H^s}} \succ^R x^{h^s_1}$.

Define $L = \prod_{s \in \Sigma} \{r^1_s, \ldots, r^{H^s}_s, q_s\}$. Note that each $l \in L$ selects, for every sequence s, one inequality: either one of the r^i_s, or q_s. If we satisfy the inequality selected by l for s then we rule out the cycle indicated by sequence s. Let J_l be the system of inequalities obtained by collecting the inequalities selected by l. Satisfaction of just one system J_l rules out all potential cycles.

We know from Afriat's Theorem that there is a solution to the system of Afriat inequalities iff there is l such that the observed data (which are the coefficients of the system of Afriat inequalities) satisfy the corresponding J_l. Thus Afriat's Theorem illustrates an instance of quantifier elimination, as formalized in Theorem 9.7.

[3] Formally, to apply the theorem, we need to treat $a \neq 0$ as an inequality in the original system, although the existentially quantified x does not appear there. The inequality $a \neq 0$ then also appears in the system in which x has been eliminated.

9.4 THE CORE OF EXCHANGE ECONOMIES

We finalize the discussion of exchange economies by briefly looking instead into the core of Walrasian equilibrium. Much less is known about the core than about equilibrium, and we need to restrict attention to the case of two agents. The material here is due to Bossert and Sprumont.

Consider an exchange economy with two consumers. Suppose that there is an aggregate endowment $\bar{\omega} \in \mathbf{R}^n_{++}$. In any allocation, consumption of agent 2 is determined by the consumption of agent 1. The set of possible endowments and consumptions of agent 1 is $E = \{x \in \mathbf{R}^n_+ : x \leq \bar{\omega}\}$ (either agent is permitted to have zero endowment). For a set of preferences $\succeq_1$ and $\succeq_2$, we define the *core*

$$\begin{aligned}
&\mathcal{C}(\succeq_1, \succeq_2, \omega) = \\
&\{x \in E : x \succeq_1 \omega_1 \text{ and } (\bar{\omega} - x) \succeq_2 (\bar{\omega} - \omega_1)\} \\
&\cap \{x \in \mathbf{R}^n_+ : \nexists y \in E \text{ such that } y \succ_1 x \text{ and } (\bar{\omega} - y) \succ_2 (\bar{\omega} - x)\}.
\end{aligned}$$

The first part of this equation specifies the usual individual rationality constraint. The second expresses Pareto optimality. In fact, it expresses weak Pareto optimality, but note that for strictly convex preferences, the notions coincide.

The question is, given a correspondence $c : E \rightrightarrows E$, when do there exist preferences $\succeq_1$ and $\succeq_2$, satisfying natural properties, such that for all $e \in E$, $c(e) = \mathcal{C}(\succeq_1, \succeq_2, e)$? The properties under consideration are continuity, strict monotonicity, and strict convexity.

We say that a set $S \subseteq E$ is *connected* if it is connected in the usual topological sense; that is, the set cannot be partitioned into nontrivial disjoint open sets. Viewing c as a correspondence, we can also define continuity in the usual sense (of the joint hypotheses of lower and upper hemicontinuity). We say that the correspondence c is *regular* if it is continuous in e, and for all e, $c(e)$ is connected.

For the core with strictly convex and monotone preferences, if some vector e is ever selected for an endowment e', then e must be the uniquely chosen element when e' is the endowment. We say that c satisfies *persistence* if for all $e, e' \in E$, if $e \in c(e')$, then $c(e) = \{e\}$.

Two more technical conditions are specified. The conditions are a method of "identifying" the best and worst elements in the correspondence, and postulating that they behave naturally with respect to the usual order on $\mathbf{R}^n$.

Define, for $e \in E$,

$$\mathcal{S}_0(e) = \{S \subseteq E : c(e) \cup \{0\} \subseteq S, S \text{ path connected }\}$$

and

$$\mathcal{S}_\omega(e) = \{S \subseteq E : c(e) \cup \{\omega\} \subseteq S, S \text{ path connected }\}.$$

We say that c satisfies *strict path monotonicity* if for all $e, e' \in E$ for which $e \geq e'$ and $e \neq e'$, we have

$$\bigcap_{S \in \mathcal{S}_0(e)} S \subsetneq \bigcap_{S \in \mathcal{S}_0(e')} S$$

and

$$\bigcap_{S \in \mathcal{S}_\omega(e')} S \subsetneq \bigcap_{S \in \mathcal{S}_\omega(e)} S.$$

Finally, we define $\mathcal{S}(e, e') = \{S \subseteq E : c(e) \cup c(e') \subseteq S, S \text{ path connected }\}$. Say that c satisfies *strict averaging reduction* if for all $e, e' \in E$ for which $e \neq e'$ and all $\lambda \in (0,1)$, $c(\lambda e + (1-\lambda)e')$ is contained in the interior of $\bigcap_{S \in \mathcal{S}(e,e')} S$ relative to $\bigcup_{e \in E} c(e)$.

Theorem 9.9 *There exist strictly convex, strictly monotonic, and continuous $\succeq_1$ and $\succeq_2$ for which $c(e) = \mathcal{C}(\succeq_1, \succeq_2, e)$ for all $e \in E$ iff $c : E \rightrightarrows E$ satisfies regularity, persistence, strict path monotonicity, and strict averaging reduction.*

9.5 CHAPTER REFERENCES

The Sonnenschein–Mantel–Debreu Theorem was first established for the case when Z is a polynomial in Sonnenschein (1972). Mantel (1974) proved a decomposition of Z as in the statement of the theorem, but using $2n$ agents. The final statement was obtained by Debreu (1974). Proposition 9.1 is due to Debreu (1974). Corollary 9.3 is proved in Mas-Colell (1977); a much more general statement appears there: that any compact subset of the interior of the price sphere can be the equilibrium price set of a well-behaved economy. The strengthening follows from a strengthening of the SMD Theorem. The result pertaining to indices can also be found in this work; see also Mas-Colell, Whinston, and Green (1995). Geanakoplos (1984) constructs an explicit utility function to rationalize the individual excess demand functions constructed in Debreu's proof. Theorem 9.2 is from McFadden, Mas-Colell, Mantel, and Richter (1974).

Theorem 9.4 is due to Mantel (1976). Mantel shows more. The endowments in the construction in the theorem can be chosen arbitrarily, as long as they are linearly independent. In particular, they can be close to proportional: when they are exactly proportional (and preferences homothetic) we know that the economy admits a representative consumer, which of course strongly limits the excess demand function. A graphical illustration of the proof of Theorem 9.4 can be found in Shafer and Sonnenschein (1982) (attributed to Mas-Colell).

The SMD Theorem has a host of important implications. Uzawa (1960b) has shown that Brouwer's Fixed-Point Theorem is equivalent to the existence theorem of the zeroes of a continuous function satisfying the properties usually obtained from exchange economies. The SMD Theorem implies that the Fixed-Point Theorem is equivalent to a standard equilibrium existence theorem. In recent years, this has been exploited by computer scientists studying the computational complexity of finding competitive equilibria (see Papadimitriou and Yannakakis (2010); this possibility was anticipated by Mantel (1977)).

Theorem 9.6, the example in 9.3, and the idea of using the Tarski–Seidenberg result appear in Brown and Matzkin (1996). The Tarski–Seidenberg result is due to Tarski (1951), and was popularized by Seidenberg (1954). The computational complexity of quantifier elimination in this environment was described in Davenport and Heintz (1988). Brown and Shannon (2000) show that, in the setting assumed by Brown and Matzkin, certain regularity conditions of equilibria (that all equilibria are locally stable, and that the equilibrium correspondence is monotone) have no testable implications in addition to Walrasian rationalizability.

The discussion in 9.4, and in particular, Theorem 9.9, is due to Bossert and Sprumont (2002).

We have not discussed the literature on public goods. Snyder (1999) provides a discussion of the relevant ideas.

The survey paper by Carvajal, Ray, and Snyder (2004) covers in depth some of the same material as this chapter.