

# CONFIRMATORY TETRAD ANALYSIS

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*A “tetrad” refers to the difference in the products of certain covariances (or correlations) among four random variables. A structural equation model often implies that some tetrads should be zero. These “vanishing tetrads” provide a means to test structural equation models. In this paper we develop confirmatory tetrad analysis (CTA). CTA applies a simultaneous test statistic for multiple vanishing tetrads developed by Bollen (1990). The simultaneous test statistic is available in asymptotically distribution-free or normal-distribution versions and applies to covariances or to correlations. We also offer new rules for determining the nonredundant vanishing tetrads implied by a model and develop a method to estimate the power of the statistical test for vanishing tetrads. Testing vanishing tetrads provides a test for model fit that can lead to results different from the usual likelihood-ratio (LR) test associated with the maximum likelihood methods that dominate the structural equation field. Also, the CTA technique applies to some underidentified models. Furthermore, some models that are not nested according to the traditional LR test are nested in terms of vanishing tetrads. Finally, CTA does not require numerical minimization and thus avoids the associated convergence problems that are present with other estimation approaches.*

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In 1904 Spearman laid the groundwork for what was to become factor analysis. In this and more clearly in his later work (e.g., Spearman 1927), he demonstrated that a single factor underlying four or more observed variables implies that the difference in the products of certain pairs of the covariances (or correlations) of these variables must be zero. These came to be referred to as “vanishing tetrads.” The use of vanishing tetrads to examine models with latent variables dominated the work on factor analysis for the first third of the twentieth century. This approach eventually gave way to other techniques such as principal components (Hotelling 1933) and later to the maximum likelihood (e.g., Lawley and Maxwell 1971) and weighted least squares (e.g., Browne 1984) estimators that dominate today’s factor analyses. The general structural equation models (SEM) that have now swept through most of the social sciences initially also flirted with the tetrad approach to model testing (e.g., Costner 1969; Duncan 1972; Kenny 1974), but it has been replaced by the maximum likelihood method popularized by Jöreskog (1973) in the LISREL program (Jöreskog and Sörbom 1989).

The tetrad approach to SEM was all but forgotten until Glymour et al. (1987) proposed vanishing tetrads as a viable method to search for models that are consistent with the covariance matrix of observed variables. Their emphasis has been exploratory tetrad analysis (ETA) based on a computer intensive search algorithm to formulate models with a good match to the tetrads of the observed variables. In this paper we propose a confirmatory tetrad analysis (CTA) that tests one or several specific models. CTA is “confirmatory” in that models are specified in advance. The structure of each model often implies population tetrads that should be zero. A test of a model’s vanishing tetrads is a test of the model’s fit. Significant nonzero tetrads for the model implied vanishing tetrads cast doubt on the appropriateness of the model.

The relation between ETA and CTA is analogous to that between exploratory and confirmatory factor analysis. Our CTA approach differs from Glymour et al.’s ETA in several ways. First, CTA is meant to *test* rather than to generate models, the latter being the purpose of ETA. As such, CTA and ETA are not rival methods. Second, we employ a simultaneous test for vanishing tetrads that applies to normal or nonnormal variables (Bollen 1990). Glymour et al. (1987) use Wishart’s (1928) *single* tetrad test that assumes

multinormally distributed observed variables and that does not control for multiple testing problems. Third, we also provide a modification for the test statistic so that it applies to correlations as well as covariances. Fourth, we look only at the nonredundant vanishing tetrads whereas Glymour et al. (1987) examine all vanishing tetrads.

A natural question is why should we consider CTA when we already have confirmatory factor analysis and the other maximum likelihood (ML)/weighted least squares (WLS) approaches to the general SEM? There are several good reasons. First, testing vanishing tetrads provides a goodness-of-fit test for a model that can lead to results different from the usual likelihood-ratio (LR) test associated with the ML/WLS methods.<sup>1</sup> We do not claim that our test is superior to the LR test, but it may be possible to reveal specification errors that are not evident in the LR test. Second, the CTA technique applies to some underidentified models. We can have a test of model fit even if the parameters of the model cannot be uniquely determined.<sup>2</sup> Third, some models that are not nested according to the conventional LR test are nested in terms of vanishing tetrads.<sup>3</sup> CTA allows the overall fit of some "nonnested" models to be compared directly. Finally, as mentioned previously, we have asymptotically distribution-free tests that apply to covariances and correlations. Although distribution-free estimators also are available for SEM through the work of Browne (1984) and others, the main advantage of our technique is that CTA uses a

<sup>1</sup>We use the term "likelihood ratio (LR)" test here and throughout the paper to refer to the tests that are based on ML estimation as well as on WLS estimation. Strictly speaking, the LR test refers only to the test statistic from ML methods. However, Browne (1984), among others, justifies the usual ML fitting functions and test statistics under the less restrictive WLS family of estimators. Thus, for the sake of brevity, we use LR test to mean the overall fit tests derived from ML or WLS methods.

<sup>2</sup>Shapiro (1986) discusses the theoretical conditions where it is possible to have an LR or WLS test statistic that follows an asymptotic chi-square distribution for some underidentified models. We know of no empirical applications of this work.

<sup>3</sup>It is possible to compare the fit of nonnested models using some of the overall fit measures in structural equation models that take the degrees of freedom of a model into account (see, for example, Bollen 1989, pp. 256–81). However, these typically do not provide a test of the statistical significance of the differences in fit for nonnested models. A growing literature on significance testing for nonnested models is accumulating (e.g., MacKinnon 1983; Judge et al. 1985, pp. 881–85), but little of this work has penetrated the structural equation literature. On the other hand, this literature on nonnested models has not considered vanishing tetrads as a test for such models.

noniterative estimator that does not have nonconvergence problems as is sometimes true for the commonly used procedures.

In this paper we propose CTA and illustrate its application to the previously mentioned issues. We view CTA not as a replacement for the standard methods of SEM but rather as a technique that complements current methods of model evaluation. For models that are not easily testable under the conventional methods, CTA sometimes can be a useful tool for model evaluation. In the following sections, we will discuss the concept of vanishing tetrads, propose new rules for selecting nonredundant vanishing tetrads, provide a method of significance testing, and develop a method to estimate the power of the vanishing tetrad test. Finally, we will illustrate the applications of CTA with examples.

## 1. MODEL IMPLIED VANISHING TETRADS

The idea of vanishing tetrads is best introduced by way of examples. Figure 1(a) is a path diagram of a factor model with one latent variable ( $\xi_1$ ) and four observed variables ( $x_1$  to  $x_4$ ). We use the usual path analysis conventions where an oval or circle signifies a latent variable and boxes denote observed variables. Disturbances (or errors) are not enclosed. A single-headed straight arrow indicates an effect of the variable from the base of the arrow to the variable at the head of the arrow. The equations corresponding to this diagram are of the form:

$$x_i = \lambda_{i1}\xi_1 + \delta_i \quad (1)$$

where  $\delta_i$  is the random measurement error (disturbance) term with  $E(\delta_i) = 0$  for all  $i$ ,  $\text{COV}(\delta_i, \delta_j) = 0$  for  $i \neq j$ , and the  $\text{COV}(\xi_1, \delta_i) = 0$  for all  $i$ . All variables are written as deviations from their means to simplify the algebra.

The population covariances ( $\sigma_{ij}$ ) of the observed variables are of the following form:

$$\sigma_{ij} = \lambda_{i1}\lambda_{j1}\phi, \quad (2)$$

where  $\sigma_{ij}$  is the population covariance of the  $i$  and  $j$  variables and  $\phi$  is the variance of  $\xi_1$ . If the model is correct, then we can use covariance algebra (e.g., Bollen 1989, p. 21) to prove that the equalities below must hold:

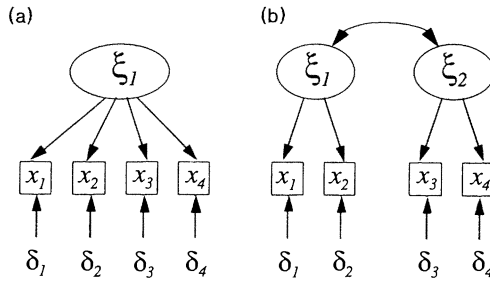


FIGURE 1. Two factor models.

$$\begin{aligned}
 \tau_{1234} &= \sigma_{12}\sigma_{34} - \sigma_{13}\sigma_{24} = 0 \\
 \tau_{1342} &= \sigma_{13}\sigma_{42} - \sigma_{14}\sigma_{32} = 0 \\
 \tau_{1423} &= \sigma_{14}\sigma_{23} - \sigma_{12}\sigma_{43} = 0,
 \end{aligned}
 \tag{3}$$

where  $\tau_{ghij}$  is the population tetrad difference that equals the quantity to its right. We use the Kelley (1928) notation for tetrads, where  $\tau_{ghij}$  refers to  $\sigma_{gh}\sigma_{ij} - \sigma_{gi}\sigma_{hj}$ . When  $\tau_{ghij}$  is zero for a model, this is referred to as a vanishing tetrad. The model in Figure 1(a) implies the three vanishing tetrads in equation (3). Due to sampling errors, the sample counterpart,  $t_{ghij}$ , is likely to be nonzero. A simultaneous significance test described later can be used to determine whether the model in Figure 1(a) is consistent with the sample data. A nonsignificant test statistic means that the implied vanishing tetrads hold and the model is a legitimate candidate for consideration. If the significance test indicates otherwise, the one-factor model in Figure 1(a) would be rejected.

Figure 1(b) shows a two-factor model with two indicators for each latent variable. The only vanishing tetrad implied by this model is

$$\tau_{1342} = \sigma_{13}\sigma_{42} - \sigma_{14}\sigma_{32} = 0.
 \tag{4}$$

A significance test of this vanishing tetrad provides a test of the model in Figure 1(b). Notice that the vanishing tetrad implied by the model in Figure 1(b) [see equation (4)] is a subset of the vanishing tetrads implied by the model in Figure 1(a) [see equation (3)]. Whenever the vanishing tetrads of one model are a subset of those in

another, we refer to such models as having "nested tetrads." If the difference in the test statistics for the two models is not significant, this lends support to the model that implies the most vanishing tetrads. If the test result is significant, we would prefer the model with the fewest vanishing tetrads. In Figure 1 we would favor the one factor model if the test statistic for the vanishing tetrads in equation (3) is not significantly greater than the test statistic for the vanishing tetrads in equation (4).

## 2. IDENTIFYING VANISHING TETRADS

To perform significance tests, we need to identify the vanishing tetrads implied by a model. We propose three methods for this task: covariance algebra, a new rule for factor analysis models, and a new empirical method for general SEM.<sup>4</sup>

### 2.1. *Covariance Algebra*

The first method uses covariance algebra to show the vanishing tetrads for a model. The starting point is the structural equations and assumptions for a model (for example, see equation [1]). A few simple rules of covariance algebra (Bollen 1989, p. 21) allow us to express the covariance of any two variables in terms of the parameters of the model (for example, see equation [2]). A more general way of obtaining the covariances of the observed variables in terms of the model parameters is to use matrix methods to form the model implied covariance matrix for a model (see Jöreskog and Sörbom 1989, p. 5). We can then compare two pairs of covariances in a tetrad and conclude whether a vanishing tetrad is implied by the model. Whether a vanishing tetrad is implied does not depend on the value of the coefficients unless one or more have a trivial zero coefficient or the unlikely coincidence occurs that the combination of values of the parameters lead to zero. In other words, in practice the structure of a model determines the vanishing tetrads, not the specific values of the parameters.

<sup>4</sup>Another possibility is to use Glymour et al. (1987) computer algorithms to determine the vanishing tetrads of models.

## 2.2. A Factor Analysis Rule

Using covariance algebra, as we did for the models in Figure 1, becomes tedious for models with more than four variables. The second method, which can simplify the task, works for factor analysis models where each indicator is influenced only by one latent variable and an error variable, though this rule permits correlated errors of measurement. A vanishing tetrad is implied when two conditions are met: (1) none of the four covariances in a tetrad equation involve correlated error terms and (2) the two pairs of latent variables associated with the two covariances in the first term match those in the second term of the equation.

Regardless of the size of the model, we consider four variables at a time and repeat the process for every foursome of variables in the model. For every four variables, there are three possible vanishing tetrads, and each of them has to be checked regarding whether it fulfills the above two conditions. Suppose  $x_1, x_2, x_3, x_4$  are four indicators in a factor model with each observed indicator affected only by one latent variable and an error variable. The four measurement equations are:

$$\begin{array}{ll} x_1 = \lambda_{1i}\xi_i + \delta_1, & \text{and} \quad x_3 = \lambda_{3k}\xi_k + \delta_3, \\ x_2 = \lambda_{2j}\xi_j + \delta_2, & x_4 = \lambda_{4l}\xi_l + \delta_4. \end{array}$$

For instance, whether  $\tau_{1234} = \sigma_{12}\sigma_{34} - \sigma_{13}\sigma_{24} = 0$  is implied by a model depends on  $\sigma_{12}\sigma_{34}$  and  $\sigma_{13}\sigma_{24}$ . Each correlated error of  $\text{COV}(\delta_1, \delta_2)$ ,  $\text{COV}(\delta_3, \delta_4)$ ,  $\text{COV}(\delta_1, \delta_3)$ , and  $\text{COV}(\delta_2, \delta_4)$  has a unique effect on the covariance of  $\sigma_{12}$ ,  $\sigma_{34}$ ,  $\sigma_{13}$ , and  $\sigma_{24}$  respectively. If any of the four correlated error terms is nonzero,  $\sigma_{12}\sigma_{34}$  will not equal  $\sigma_{13}\sigma_{24}$ , except under the very unlikely case where the effects of correlated errors on the four covariances cancel each other out. A vanishing tetrad is implied only under the condition that none of the four covariances involves a correlated error term. (This condition can be used to rule out vanishing tetrads in models other than the one described here.)

If we assume that the latent variables do not correlate with the error terms and the correlated errors of  $\text{COV}(\delta_1, \delta_2)$ ,  $\text{COV}(\delta_3, \delta_4)$ ,  $\text{COV}(\delta_1, \delta_3)$ , and  $\text{COV}(\delta_2, \delta_4)$  are zero, then  $\sigma_{12}\sigma_{34}$  and  $\sigma_{13}\sigma_{24}$  equal the following:

$$\sigma_{12}\sigma_{34} = \lambda_{1i}\lambda_{2j}\lambda_{3k}\lambda_{4l}\phi_{ij}\phi_{kl} \quad \text{and} \quad \sigma_{13}\sigma_{24} = \lambda_{1i}\lambda_{2j}\lambda_{3k}\lambda_{4l}\phi_{ik}\phi_{jl}.$$

It becomes obvious that  $\sigma_{12}\sigma_{34} - \sigma_{13}\sigma_{24} = 0$  iff  $\phi_{ij}\phi_{kl} = \phi_{ik}\phi_{jl}$ , provided that none of the  $\lambda_{ij}$  is zero. This equality holds only if a vanishing tetrad satisfies the second condition that the two pairs of latent variables associated with  $\sigma_{12}\sigma_{34}$  match with those associated with  $\sigma_{13}\sigma_{24}$ . In this example,  $\xi_j = \xi_k$  is necessary and sufficient to make  $\phi_{ij}\phi_{kl} = \phi_{ik}\phi_{jl}$ .

We illustrate this rule with the four examples in Figure 2. In Figure 2(a), we add a correlated error between  $x_3$  and  $x_4$  to the two-factor model in Figure 1(b). Figure 2(a) becomes an underidentified model. The vanishing tetrad,  $\tau_{1342} = 0$ , is implied in Figure 1(b) and continues to be true in Figure 2(a). The correlated error term that appears in  $\sigma_{34}$  has no effects on this tetrad equation, and the two pairs

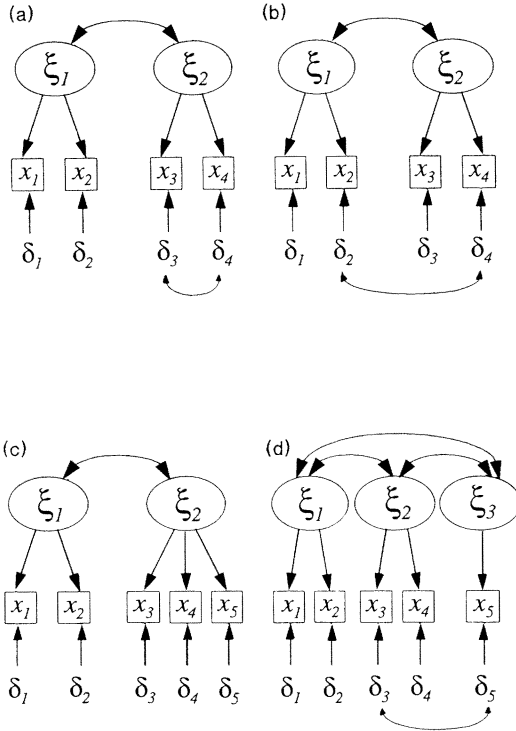


FIGURE 2. Four factor models.

of latent variables,  $(\xi_1, \xi_2)$  and  $(\xi_2, \xi_1)$ , of  $\sigma_{13}\sigma_{42}$  match those of  $\sigma_{14}\sigma_{32}$ . This vanishing tetrad is no longer implied in Figure 2(b), however, because a correlated error term appears in  $\sigma_{24}$ . The other two tetrad equations,  $\tau_{1234} = 0$  and  $\tau_{1423} = 0$ , are not implied in the two-factor models in Figure 2(a) and 2(b) because the corresponding pairs of latent variables in the first and the second terms of the tetrad equations do not match each other. The correlated error term in Figure 2(a) alone is sufficient to rule out these two vanishing tetrads. Consequently, one tetrad is implied in Figure 2(a) and none is implied in Figure 2(b). As such, we have nested tetrads and we can compare the two models. Note that these two models are not nested in terms of an LR test, though they are nested for a tetrad test.

The same procedure applies to models with more variables. With five variables, such as the models in Figures 2(c) and 2(d), we have five sets  $(5!/1!4!)$  of tetrad equations. The task in Figure 2(c) is simplified because the model is composed of only two basic structures. Consider  $x_1, x_2, x_3$ , and  $x_4$  with two indicators for each latent variable. This part of the model is identical to the model in Figure 1(b), and it implies the same vanishing tetrad,  $\tau_{1342}$ . In an analogous fashion we can find the vanishing tetrads for  $x_1, x_2, x_3$ , and  $x_5$ , and for  $x_1, x_2, x_4$ , and  $x_5$  since they share the same basic structure of two indicators per latent variable.

The second basic structure has one indicator for  $\xi_1$  and three indicators for  $\xi_2$ . Consider  $x_1, x_3, x_4$ , and  $x_5$ . All three vanishing tetrads ( $\tau_{1345} = 0$ ,  $\tau_{1453} = 0$ , and  $\tau_{1534} = 0$ ) are implied. First, no correlated errors exist in any of the covariances, and second,  $(\xi_1, \xi_2)$  and  $(\xi_2, \xi_2)$  are the two pairs of latent variables in the first and the second terms of each tetrad equation. Similarly, the three vanishing tetrads among  $x_2, x_3, x_4$ , and  $x_5$  are also implied for the same reasons.

We modify Figure 2(c) by adding one more latent variable and one correlated error term in Figure 2(d). The rule for determining implied vanishing tetrads is no different from that used in the previous three examples. If we consider  $x_1$  to  $x_4$ , the model structure is identical to the one in Figure 1(b), and only  $\tau_{1342} = 0$  is implied. For  $x_1, x_2, x_3$ , and  $x_5$ ,  $\tau_{1235} = 0$  and  $\tau_{1523} = 0$  are not implied because  $\sigma_{35}$ , which is in both equations, has a correlated error term. The two pairs of latent variables associated with  $\sigma_{13}\sigma_{52}$  and  $\sigma_{15}\sigma_{32}$  are  $(\xi_1, \xi_2)$  and  $(\xi_3, \xi_1)$  and  $(\xi_1, \xi_3)$  and  $(\xi_2, \xi_1)$  respectively, and no correlated error term

appears in the four covariances in this tetrad equation. As such,  $\tau_{1352} = 0$  is implied in the model. Among  $x_1$ ,  $x_2$ ,  $x_4$ , and  $x_5$ , only  $\tau_{1452} = 0$  is implied. Finally, application of our general rule to the set of  $x_1$ ,  $x_3$ ,  $x_4$ , and  $x_5$  and the set of  $x_2$  to  $x_5$  shows no vanishing tetrads.

The same strategy for determining vanishing tetrads applies to other factor analysis models where each indicator is influenced by a single latent variable and an error term.

### *2.3. An Empirical Method*

The covariance algebra technique for determining vanishing tetrads is perfectly general but too tedious to implement for complex models. The factor analysis rule is inapplicable to models with factor complexity greater than one or to general SEM. In this subsection we describe a simple but new empirical means to determine model implied vanishing tetrads. The procedure has four steps:

1. Arbitrarily specify the values of model parameters.
2. Use model parameters specified in step 1 to generate the implied covariance matrix through structural equation programs such as LISREL (Jöreskog and Sörbom 1989), EQS (Bentler 1989), or CALIS (Hartmann 1991).
3. Calculate all tetrads.
4. Take those tetrads within rounding of zero as the model implied vanishing tetrads.

In step 1 we recommend use of the parameter estimates for a model, if available, since the implied covariance matrix for step 2 is readily accessible in the above-mentioned programs. The essence of this method is to generate a covariance matrix that is consistent with the model so that when you calculate the tetrads, those that should be zero will be within rounding error of zero. Researchers having any doubt regarding whether a value is zero or not can apply the covariance algebra method to the specific tetrads that are in question as an additional check. Our experience suggests that this method is extremely accurate. Coupled with its generality, this makes it the method of choice for most models. We will illustrate the procedure in the examples section.

### 3. REDUNDANT VANISHING TETRADS

Previous tetrad analyses, such as those of Glymour et al. (1987), focused on tests of individual vanishing tetrads; redundancy was rarely a concern except in the simple case where all three vanishing tetrads are implied by a set of four variables. As a result, there is no guidance on how to select nonredundant vanishing tetrads among all those implied by a model. For a simultaneous test of a set of implied vanishing tetrads, we have to determine which vanishing tetrads are redundant and should be excluded from the test. Otherwise, the covariance matrix of the tetrads that is part of the test statistic can be singular, and its inverse will not exist. In the material that follows we develop a procedure to deal with this problem.<sup>5</sup>

Algebraic substitution between vanishing tetrads will show that some of the vanishing tetrads can be derived from the others and are redundant for the test. When none of the covariances are in common between vanishing tetrads, algebraic substitution is impossible. When two vanishing tetrads have three or more covariances in common, they must be identical. Therefore, we need to consider only two cases: those vanishing tetrads having either one or two covariances in common.

When two covariances in one vanishing tetrad are identical with the covariances in another vanishing tetrad, it is a sufficient condition that a third vanishing tetrad must be implied and should be eliminated in the simultaneous test. The simplest case is when all three vanishing tetrads are implied for a set of four variables, only two of them are needed for the simultaneous test due to redundancy. For instance, if

$$\begin{aligned}\tau_{abcd} &= \sigma_{ab}\sigma_{cd} - \sigma_{ac}\sigma_{bd} = 0, \\ \tau_{acdb} &= \sigma_{ac}\sigma_{db} - \sigma_{ad}\sigma_{cb} = 0, \quad \text{and} \\ \tau_{adb c} &= \sigma_{ad}\sigma_{bc} - \sigma_{ab}\sigma_{cd} = 0,\end{aligned}$$

then any two of them imply the third—that is, only two vanishing tetrads are independent. Suppose we have two vanishing tetrads

<sup>5</sup>In comments on this paper, Yu Xie suggested a method for determining the nonredundant vanishing tetrads by using an analogy to methods of determining the odds-ratios in a contingency table. However, the method was suggested for single factor models without correlated errors of measurement, and it is not clear whether the procedure generalizes to other models.

$$\tau_{acdb} = \sigma_{ac}\sigma_{db} - \sigma_{ad}\sigma_{cb} = 0 \text{ and} \quad (5)$$

$$\tau_{adeb} = \sigma_{ad}\sigma_{eb} - \sigma_{ae}\sigma_{db} = 0, \quad (6)$$

where  $\sigma_{ad}$  and  $\sigma_{db}$  appear in both vanishing tetrads. Algebraic manipulation between (5) and (6) will show that

$$\tau_{aceb} = \sigma_{ac}\sigma_{eb} - \sigma_{ae}\sigma_{cb} = 0 \quad (7)$$

is implied.

In the case where there is only one common covariance between two vanishing tetrads, algebraic substitution will lead to a vanishing equation with six covariances, and no additional vanishing tetrad will be implied. For example,

$$\tau_{abcd} = \sigma_{ab}\sigma_{cd} - \sigma_{ac}\sigma_{bd} = 0 \text{ and} \quad (8)$$

$$\tau_{abef} = \sigma_{ab}\sigma_{ef} - \sigma_{ae}\sigma_{bf} = 0 \quad (9)$$

imply

$$\sigma_{ac}\sigma_{bd}\sigma_{ef} - \sigma_{ae}\sigma_{bf}\sigma_{cd} = 0. \quad (10)$$

Introducing more vanishing tetrads with one common covariance with equation (10) only will further expand the equation. The single possibility is to have another vanishing tetrad that has three covariances in common with equation (10) such that three covariances can be eliminated and a new covariance term will be added to equation (10). Consider

$$\tau_{aecd} = \sigma_{ae}\sigma_{cd} - \sigma_{ac}\sigma_{ed} = 0. \quad (11)$$

In vanishing tetrad (11),  $\sigma_{ac}$ ,  $\sigma_{ae}$ , and  $\sigma_{cd}$  appear in equation (10). Equations (10) and (11) together imply a redundant vanishing tetrad,

$$\tau_{bfde} = \sigma_{bf}\sigma_{de} - \sigma_{bd}\sigma_{fe} = 0. \quad (12)$$

That means given vanishing tetrads (8), (9), and (11), vanishing tetrad (12) should be excluded in the simultaneous test.

Alternatively, with the rule of two common covariances, vanishing tetrad (12) can be concluded from pairwise algebraic substitution between vanishing tetrads. Notice that  $\sigma_{ab}$ ,  $\sigma_{ac}$ ,  $\sigma_{ae}$ , and  $\sigma_{cd}$  appear twice in vanishing tetrads (8), (9), and (11). These four covariances can be eliminated through algebraic substitution and the remaining four covariances form vanishing tetrad (12). We begin with vanishing

tetrads (8) and (11) because  $\sigma_{ac}$  and  $\sigma_{cd}$  appear in both equations. As a result, another vanishing tetrad

$$\tau_{aebd} = \sigma_{ae}\sigma_{bd} - \sigma_{ab}\sigma_{ed} = 0, \quad (13)$$

is implied. In vanishing tetrads (9) and (13), both have  $\sigma_{ab}$  and  $\sigma_{ae}$ , and (9) and (13) together lead to the redundant vanishing tetrad (12). This example illustrates that pairwise comparisons between those vanishing tetrads with two common covariances are an adequate means for identifying redundant vanishing tetrads.

The above example shows that vanishing tetrads (8), (9), (11), (12), and (13) are linearly dependent among each other; only three of them are needed for model testing. If the model is correct and the null hypothesis is true, then the choice of the three vanishing tetrads matters little. With an incorrect model and a false null hypothesis, it is possible that the selection might matter more. In our experience with the examples in the empirical example section, we found similar results regardless of the choice of the nonredundant vanishing tetrads. However, as a precaution one could select a different set of redundant vanishing tetrads to exclude and recalculate the test of significance. Since more than one significance test is being performed, the researcher should adjust the individual alpha levels for the significance tests to maintain an overall alpha level for the family of tests. A Bonferroni correction is probably the easiest one to implement. Consistent test results increase our confidence in the initial results while inconsistent test results indicate that the model is not correct.<sup>6</sup>

#### 4. SIGNIFICANCE TESTING OF VANISHING TETRADS

Spearman and Holzinger (1924), Kelley (1928), Wishart (1928), and Kenny (1974) have proposed significance tests for a vanishing tetrad. All these tests are asymptotic, assume a multivariate normal distribution among the observed variables, and are not simultaneous tests for multiple vanishing tetrads. Bollen (1990) proposed a less restrictive

<sup>6</sup>There are two other possible sensitivity checks: (1) Take the pool of redundant tetrads and perform the simultaneous significance test on them, after eliminating any redundant tetrads in this group, and (2) perform individual vanishing tetrad tests on the redundant tetrads to see if any are significant. Use a Bonferroni correction to take into account the multiple tests that are performed.

test that evaluates multiple tetrads simultaneously, applies to normally or nonnormally distributed observed variables, and analyzes correlations or covariances. This test was originally proposed for ETA but is applicable to CTA as well. The null hypothesis is  $H_0: \tau = \mathbf{0}$ , and the alternative hypothesis is  $H_a: \tau \neq \mathbf{0}$  where  $\tau$  is a vector of the population tetrads that are implied to be zero for a specific model. A significant test statistic suggests that the model implied vanishing tetrads are not zero and casts doubt on the model's validity.

To derive the test statistic, we begin with a vector  $\mathbf{S}$  that includes the nonredundant elements of  $\mathbf{S}$ , the unbiased sample covariance matrix of the observed variables.<sup>7</sup> Let  $\sigma$  be a similar vector formed from  $\Sigma$ , the population covariance matrix of the observed variables. We assume that the fourth-order moments of the observed variables exist and are finite. The  $E(\mathbf{s})$  is  $\sigma$ . The distribution of  $\sqrt{N}(\mathbf{s} - \sigma)$  in finite samples is not always known but the limiting distribution is multivariate normal with a mean of zero and a covariance matrix of  $\Sigma_{ss}$  (e.g., see Browne 1984, p. 64):

$$\sqrt{N}(\mathbf{s} - \sigma) \xrightarrow{D} N(\mathbf{0}, \Sigma_{ss}). \quad (14)$$

The elements of  $\Sigma_{ss}$  give the variances and covariances of the sample covariances. In general the elements of  $\Sigma_{ss}$  equal

$$[\Sigma_{ss}]_{ef,gh} = \sigma_{efgh} - \sigma_{ef} \sigma_{gh}, \quad (15)$$

where  $\sigma_{efgh}$  is the fourth-order moment for the  $e, f, g$ , and  $h$  variables. A sample estimator of  $\sigma_{efgh}$  is

$$s_{efgh} = N^{-1} [\Sigma (X_e - \bar{X}_e) (X_f - \bar{X}_f) (X_g - \bar{X}_g) (X_h - \bar{X}_h)]. \quad (16)$$

If the observed variables are multinormally distributed, then the elements of  $\Sigma_{ss}$  are

$$\sigma_{eg} \sigma_{fh} + \sigma_{eh} \sigma_{fg}. \quad (17)$$

Instead of the asymptotic covariance matrix of the sample covariances, we require the asymptotic variance of the sample tetrad differences. Define  $\mathbf{t}$  as the column vector of the independent sample tetrad differences implied by a model,  $\tau(\sigma)$  as the column vector of the population vanishing tetrads that is a function of  $\sigma$ , and  $\sigma$  as the

<sup>7</sup>The derivation of this test statistic is based on the description in Bollen (1990).

column vector of all  $\sigma_{ef}$  that appear in one or more of the vanishing tetrads. The tetrad differences,  $\mathbf{t}$ , are nonlinear functions of the sample covariances. Assume that  $\boldsymbol{\pi}(\boldsymbol{\sigma})$  is a continuously differentiable function with respect to  $\boldsymbol{\sigma}$  in a neighborhood of the true value of  $\boldsymbol{\sigma}$ , say  $\boldsymbol{\sigma}_o$ , that does not vanish at  $\boldsymbol{\sigma}_o$ . In conjunction with equation (14), we can use the delta method (Rao 1973, 385–89; Bishop, Fienberg, and Holland 1975, 486–500) to estimate the asymptotic variance of  $\mathbf{t}$ . Using this theorem, we have

$$\sqrt{N}\mathbf{t} \xrightarrow{D} N(\mathbf{0}, \boldsymbol{\Sigma}_t) \quad (18)$$

$$\boldsymbol{\Sigma}_t = (\partial \boldsymbol{\pi} / \partial \boldsymbol{\sigma})' \boldsymbol{\Sigma}_{ss} (\partial \boldsymbol{\pi} / \partial \boldsymbol{\sigma}), \quad (19)$$

where  $\boldsymbol{\Sigma}_t$  is the covariance matrix of the limiting distribution of the sample tetrad differences and  $\boldsymbol{\Sigma}_{ss}$  is the covariance matrix of the limiting distribution of the sample covariances that appear in the sample tetrad differences. Assume that  $\boldsymbol{\Sigma}_{ss}$  is continuous with respect to the fourth order moments and elements of  $\boldsymbol{\sigma}$  of which it is a function in a neighborhood of the true values of the fourth order moments and  $\boldsymbol{\sigma}_o$ . Then all the parameters in (19) can be estimated by replacing the population parameters by their sample counterparts. Note also that this can be made a distribution-free estimator of the asymptotic covariance matrix by the choice of  $\boldsymbol{\Sigma}_{ss}$ . The main diagonal of  $\boldsymbol{\Sigma}_t$  contains the variances of the sample tetrad differences while the off-diagonal elements contain their covariances for the limiting distribution.

A test statistic of whether all tetrad differences are zero is

$$T = N \mathbf{t}' \hat{\boldsymbol{\Sigma}}_t^{-1} \mathbf{t}. \quad (20)$$

Asymptotically,  $T$  approximates a chi-square variate with  $df$  equal to the number of tetrad differences simultaneously examined. The  $H_o$  is that all tetrad differences implied by a model are zero (i.e.,  $\boldsymbol{\tau} = \mathbf{0}$ ). Failure to reject  $H_o$  provides support for the model whereas rejection suggests that one or more tetrad differences are different from zero. When there is only one tetrad difference in  $\mathbf{t}$ , then (20) equals:

$$(t_{1432})^2 / \text{AVAR}(t_{1432}). \quad (21)$$

Also, the test statistic generalizes to hypotheses of nonzero values of  $\boldsymbol{\tau}$  by replacing  $\mathbf{t}$  with  $(\mathbf{t} - \boldsymbol{\tau})$  in equation (20), with  $\boldsymbol{\tau}$  containing the values of the population tetrads under  $H_o$ .

The results can be modified to apply to tetrad differences of correlation coefficients rather than covariances. The key change is to replace the covariance matrix of the covariances (i.e.,  $\Sigma_{ss}$ ) with the covariance matrix of the correlation coefficients (i.e.,  $\Sigma_{rr}$ ). The elements of  $\Sigma_{rr}$  for arbitrary distributions are (Isserlis 1916)

$$\begin{aligned} [\Sigma_{rr}]_{ef,gh} = & \rho_{efgh} + (1/4)\rho_{ef}\rho_{gh}(\rho_{eegg} + \rho_{ffgg} + \rho_{eehh} + \rho_{ffhh}) \\ & - (1/2)\rho_{ef}(\rho_{eegh} + \rho_{ffgh}) \\ & - (1/2)\rho_{gh}(\rho_{efgg} + \rho_{efhh}), \end{aligned} \quad (22)$$

where  $\rho_{efgh}$  is the standardized fourth order moment and  $\rho_{ef}$  is the population correlation of variable  $e$  and  $f$ .

For a multinormal distribution, this simplifies to

$$\begin{aligned} [\Sigma_{rr}]_{ef,gh} = & (1/2)\rho_{ef}\rho_{gh}(\rho_{eg}^2 + \rho_{eh}^2 + \rho_{fg}^2 + \rho_{fh}^2) \\ & + \rho_{eg}\rho_{fh} + \rho_{eh}\rho_{fg} - \rho_{ef}(\rho_{fg}\rho_{fh} + \rho_{eg}\rho_{eh}) \\ & - \rho_{gh}(\rho_{fg}\rho_{eg} + \rho_{fh}\rho_{eh}). \end{aligned} \quad (23)$$

Thus all of the above discussion applies to tetrad differences of correlations as well as of covariances.

## 5. POWER OF VANISHING TETRAD TEST

The power of a statistical test is the probability of rejecting a false null hypothesis when an alternative hypothesis is true. Recent research in SEM has provided ways to estimate the power of the chi-square likelihood ratio test of  $H_0: \Sigma = \Sigma(\theta)$ , where  $\Sigma$  is the population covariance matrix of the observed variables,  $\Sigma(\theta)$  is the covariance matrix implied by the hypothesized model, and  $\theta$  is the vector of free parameters in a model (Satorra and Saris 1985; Matsueda and Bielby 1986; Bollen 1989, pp. 338–49). It would be helpful to know the power of the simultaneous vanishing tetrad test of  $H_0: \tau = \mathbf{0}$  for several reasons. One is that with it we could determine if a significant (nonsignificant) test statistic is due to too much (or too little) power. This information would aid our assessment of a tetrad test. For instance, if we find that a vanishing tetrad has low power, yet the test statistic is highly significant, this would cast serious doubt on any model that implies the set of vanishing tetrads that were tested. Alternatively, if a tetrad test had high power and the test statistic was not statistically significant, the

plausibility of the vanishing tetrads would be increased. Second, the power estimate for the vanishing tetrad test would be helpful in the situation where conflicting results occur for the LR test and the tetrad test. Knowing the power of both tests could partially or totally explain the discrepancy.

The rationale for our method to assess the power of the vanishing tetrad test is as follows. Suppose that we replace (18) with the more general expression of

$$\sqrt{N}\mathbf{t} \xrightarrow{D} N(\boldsymbol{\tau}_a, \boldsymbol{\Sigma}_{tt}), \quad (24)$$

where  $\boldsymbol{\tau}_a$  is the column vector of the tetrads that are hypothesized to be zero for a specific model and all other symbols are defined as previously. Equation (24) equals equation (18) if we set  $\boldsymbol{\tau}_a$  to zero. However, in equation (24) we allow some or all of the population tetrads to be nonzero, an outcome that runs counter to the vanishing tetrads implied by the hypothesized model.

Under equation (24), the test statistic,  $T$ , in equation (20) asymptotically approximates a *noncentral* chi-square variate with  $df$  equal to the number of nonredundant tetrad differences simultaneously examined and with a noncentrality parameter of

$$\kappa = N \boldsymbol{\tau}_a' \boldsymbol{\Sigma}_{tt}^{-1} \boldsymbol{\tau}_a. \quad (25)$$

By knowing the  $df$ ,  $\kappa$ , and the Type I alpha level at which we test the vanishing tetrads, we can estimate the power of the simultaneous vanishing tetrad test. The  $df$  are obvious by counting the number of nonredundant vanishing tetrads implied by a model. The alpha value is the probability of a Type I error chosen by a researcher and is typically 0.05. The value of  $N$  is known and equation (19) enables us to get  $\boldsymbol{\Sigma}_{tt}^{-1}$ .<sup>8</sup> The only remaining quantity in equation (25) is  $\boldsymbol{\tau}_a$ . To get  $\boldsymbol{\tau}_a$ , we must formulate an alternative model with respect to which we are testing the power.<sup>9</sup> Give values to all of the parameters in the alternative model and form the implied covariance matrix  $[\boldsymbol{\Sigma}(\boldsymbol{\theta}_a)]$  for the observed variables at these values. Based on the tetrads that

<sup>8</sup>For variables with the same multivariate kurtosis as a normal distribution, the elements of  $\boldsymbol{\Sigma}_{ss}$  that are needed to form  $\boldsymbol{\Sigma}_{tt}$  can be taken from  $\boldsymbol{\Sigma}(\boldsymbol{\theta}_a)$ , which is the implied covariance matrix under the alternative model. For nonnormal data with excessive multivariate kurtosis, the elements of  $\boldsymbol{\Sigma}_{ss}$  can be estimated from the sample data as described in the previous section.

<sup>9</sup>The same step of formulating an alternative model with all the parameter values is necessary in the usual power tests for structural equation models.

should be zero for the hypothesized (not the alternative) model and the implied covariance matrix, we can calculate the values of  $\tau_a$ .

In brief, the steps to the procedure are:

1. Determine  $\theta_a$ , the specific values for the parameters in the alternative model.
2. Generate the implied covariance matrix,  $\Sigma(\theta_a)$ .
3. Form  $\tau_a$ , the vector of nonredundant tetrads implied under  $H_o$  using  $\Sigma(\theta_a)$  instead of  $S$ .
4. Form  $N \tau_a' \Sigma_{tt}^{-1} \tau_a$  as the noncentrality value.
5. Calculate the power of the tetrad test based on the  $df$ , the Type I probability, and the noncentrality value.

We will illustrate the procedure in the next section.

## 6. EXAMPLES

### 6.1. Example 1: Sympathy and Anger Confirmatory Factor Analysis

The first example illustrates the use of the CTA test statistic in testing the fit of a factor analysis model. Figure 3 is the path diagram for a two-factor model with each factor measured with three indicators.

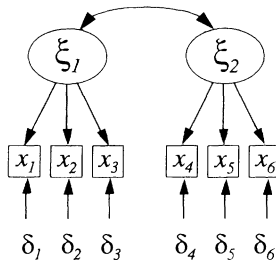


FIGURE 3. Sympathy and anger.

The data are taken from a social psychological experiment by Reisenzein (1986). As part of the experiment, he measures the feelings of sympathy and anger of 138 subjects. The covariance matrix for the six indicators is:

$$S = \begin{bmatrix} 6.982 & & & & & \\ 4.686 & 6.047 & & & & \\ 4.335 & 3.307 & 5.037 & & & \\ -2.294 & -1.453 & -1.979 & 5.569 & & \\ -2.209 & -1.262 & -1.738 & 3.931 & 5.328 & \\ -1.671 & -1.401 & -1.564 & 3.915 & 3.601 & 4.977 \end{bmatrix}$$

In Table 1, we use the sympathy and anger example of Figure 3 to illustrate the three methods of determining the vanishing tetrads. For the empirical method, the estimates of model parameters are used to generate the covariance matrix. This covariance matrix is used to calculate the tetrads. When comparing with the covariance algebra method and the factor model rules, it becomes apparent that the empirical method is effective in showing which vanishing tetrads are implied by the model. The model implied vanishing tetrads as determined by the covariance algebra and factor analysis analytic methods are virtually zero using the empirical method.

Eliminating the redundant vanishing tetrads, we identify eight independent vanishing tetrads ( $\tau_{1234} = 0, \tau_{1235} = 0, \tau_{1236} = 0, \tau_{1456} = 0, \tau_{1423} = 0, \tau_{1523} = 0, \tau_{1623} = 0$ , and  $\tau_{1645} = 0$ ). With raw data kindly provided by Reisenzein, we tested for excessive multivariate kurtosis. Mardia and Foster's (1983) "normalized" estimate of multivariate kurtosis is 6.60 (Bollen 1989, p. 424), indicating substantial positive multivariate kurtosis. The simultaneous asymptotically distribution-free test of these eight tetrads results in a chi-square of 6.71 with 8 *df* and a *p*-value of 0.57. We cannot reject the null hypothesis that all eight population tetrads are zero. Thus the sample tetrads are consistent with this model structure. Using the usual structural equation procedures, the asymptotic distribution free test statistic (weighted least squares [WLS] estimator) for this model is 6.49 with 8 *df* (*p* = 0.59), which leads to the same conclusion.

The power of these statistical tests helps in evaluating model fit. As the alternative model, we take the WLS estimates of the parameters of the original model in Figure 3 and add to it three correlated errors of measurement. The covariances of these errors are set to be equivalent to correlations of 0.1 and we generated the implied covariance matrix. Following the power procedure described in the previous section, the noncentrality parameter is 1.45 with *df* of

TABLE 1  
Model Implied Vanishing Tetrads for Sympathy and Anger Example

Tetrads	Cov. Algebra/ Factor Rules	Empirical Method
$\tau_{1234}$	implied	-0.00000000
$\tau_{1423}$	implied	-0.00000005
$\tau_{1342}$	implied	0.00000006
$\tau_{1235}$	implied	-0.00000004
$\tau_{1523}$	implied	0.00000011
$\tau_{1352}$	implied	-0.00000007
$\tau_{1236}$	implied	0.00000003
$\tau_{1623}$	implied	0.00000011
$\tau_{1362}$	implied	-0.00000014
$\tau_{1245}$		14.97524804
$\tau_{1524}$		-14.97524810
$\tau_{1452}$	implied	0.00000006
$\tau_{1246}$		14.75163923
$\tau_{1624}$		-14.75163932
$\tau_{1462}$	implied	0.00000010
$\tau_{1256}$		13.53462924
$\tau_{1625}$		-13.53462927
$\tau_{1562}$	implied	0.00000003
$\tau_{1345}$		13.99787900
$\tau_{1534}$		-13.99787907
$\tau_{1453}$	implied	0.00000008
$\tau_{1346}$		13.78886419
$\tau_{1634}$		-13.78886426
$\tau_{1463}$	implied	0.00000008
$\tau_{1356}$		12.65128313
$\tau_{1635}$		-12.65128313
$\tau_{1563}$	implied	0.00000000
$\tau_{1456}$	implied	-0.00000028
$\tau_{1645}$	implied	0.00000024
$\tau_{1564}$	implied	0.00000004
$\tau_{2345}$		10.72377520
$\tau_{2534}$		-10.72377521
$\tau_{2453}$	implied	0.00000001
$\tau_{2346}$		10.56364895
$\tau_{2634}$		-10.56364894
$\tau_{2463}$	implied	-0.00000001
$\tau_{2356}$		9.69214805
$\tau_{2635}$		-9.69214803
$\tau_{2563}$	implied	-0.00000002
$\tau_{2456}$	implied	-0.00000010
$\tau_{2645}$	implied	0.00000001
$\tau_{2564}$	implied	0.00000009
$\tau_{3456}$	implied	-0.00000007
$\tau_{3645}$	implied	0.00000003
$\tau_{3564}$	implied	0.00000003

8. With a type I error of 0.05, the power of the tetrad test is 0.11. Using the same alternative model and estimating the power of the WLS-based test statistic, we find the same 0.11 value. Thus the tetrad test and the WLS test have low power to detect the correlated errors, and the fit of the model appears less ideal than an examination of  $p$ -value for the test of null hypothesis alone would lead one to believe. Though we found the power of the tetrad test and the WLS test to be the same in this example, this will not always be the case.

### 6.2. Example 2: Union Sentiment: An SEM Without Latent Variables

The second example illustrates that CTA also applies to SEM that do not contain any latent variables. Figure 4 is the path diagram for the model taken from Bollen (1989, pp. 82–83). The data are from a study of union sentiment among southern nonunion textile workers (McDonald and Clelland 1984). The variables are deference (or submissiveness) to managers ( $y_1$ ), support for labor activism ( $y_2$ ), sentiment toward unions ( $y_3$ ), the logarithm of years in textile mill ( $x_1$ ), and age ( $x_2$ ). The sample covariance matrix ( $N = 173$ ) is (Bollen 1989, p. 120):

$$S = \begin{bmatrix} 14.610 & & & & \\ -5.250 & 11.017 & & & \\ -8.057 & 11.087 & 31.971 & & \\ -0.482 & 0.677 & 1.559 & 1.021 & \\ -18.857 & 17.861 & 28.250 & 7.139 & 215.662 \end{bmatrix}$$

The null hypothesis of multivariate normality could not be rejected for these data where the normalized test statistics for multivariate

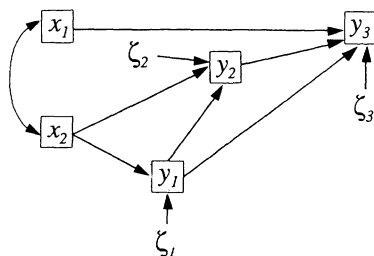


FIGURE 4. Union sentiment.

skewness was 0.74 and was  $-1.14$  for multivariate kurtosis (Bollen 1989, p.424). Thus we use the CTA and LR test statistics that are based on the normality assumption.

With either covariance algebra or our empirical method, we find the only vanishing tetrad to be  $\tau_{y_1x_1x_2y_2}$ . The CTA test statistic is 0.73 with 1 *df* ( $p = 0.39$ ). This excellent fit is consistent with the LR test statistic of 1.26 with 3 *df* ( $p = 0.74$ ).

### 6.3. Example 3: Comparison of "Nonnested" Models with Simulated Data

Some models that are not nested for the usual SEM likelihood ratio (LR) test comparison of fit are nested in the implied vanishing tetrads. The implication is that model comparisons are possible for some models that we have traditionally believed to be nonnested. We take an example from Glymour et al. (1987) to illustrate this. The three models in Figure 5 differ in the relation between the  $x_3$  and  $y_1$  variables. In Figure 5(a),  $x_3$  affects  $y_1$ , while in 5(b) the opposite relation holds. Figure 5(c) shows only correlated errors between these two variables. Clearly, from the perspective of LR test comparisons, these models are nonnested. However, the implied vanishing tetrads in Figures 5(a) and 5(b) are subsets of those implied in Figure 5(c). Figure 5(c), which has the most implied vanishing tetrads, is the most restrictive model of the three, and we can compare whether this more restrictive model fits as well as the less restrictive ones in Figures 5(a) and 5(b). We use the simulated "Data Set 2, Study 1" ( $N = 2000$ ) from Glymour et al. (1987, p. 128). The correlation matrix

$$S = \begin{bmatrix} 1 & & & & & \\ 0.73218 & 1 & & & & \\ 0.71263 & 0.61605 & 1 & & & \\ 0.65140 & 0.56910 & 0.88900 & 1 & & \\ 0.75321 & 0.65899 & 0.64722 & 0.82760 & 1 & \\ 0.69263 & 0.60523 & 0.60106 & 0.76872 & 0.87026 & 1 \end{bmatrix}$$

is generated from the model in Figure 5(c). The models in Figures 5(a) and 5(b) have 6 *df* and chi-squares of 2.76 and 3.26 ( $p$ -values of 0.84 and 0.78) respectively. The model in Figure 5(c) has 7 *df* and a chi-square value of 3.39 ( $p$ -value of 0.85). Chi-square difference tests of the first two models compared to Figure 5(c) reveal no significant differences, lending support to the validity of Figure 5(c); therefore

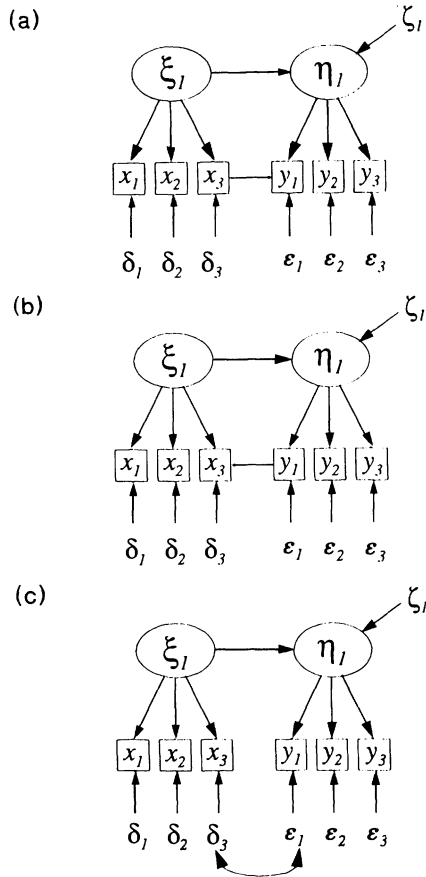


FIGURE 5. Simulated data.

we retain the model with a correlated error and select the true model. Note also that this example illustrates how the simultaneous test statistic developed here can be applied in the exploratory tetrad analyses proposed by Glymour et al. (1987) to compare alternative models that have nested vanishing tetrads.

#### 6.4. Example 4: Four-Wave Developmental Model

McArdle and Epstein (1987) introduce the path model in Figure 6 to study the developmental changes in intelligence measured by the

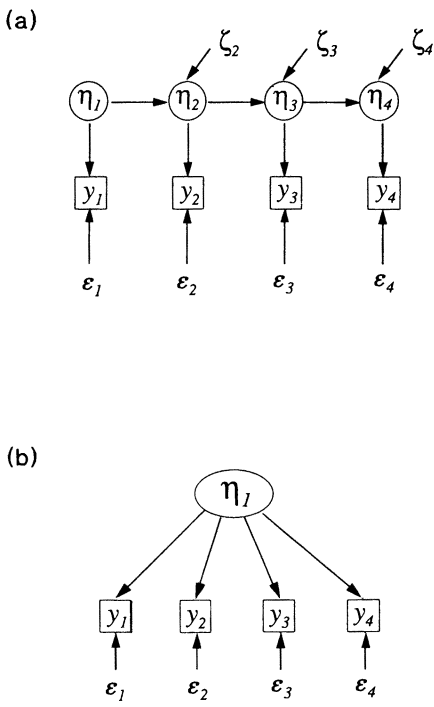


FIGURE 6. Four-wave developmental model.

Wechsler Scale in a four-wave study of 204 children. The covariance matrix is ( $N = 204$ ):

$$S = \begin{bmatrix} 40.628 & & & \\ 37.741 & 53.568 & & \\ 40.051 & 48.500 & 60.778 & \\ 50.643 & 63.169 & 70.200 & 107.869 \end{bmatrix}$$

The model has four latent variables, each of which has one indicator. Each latent variable is determined only by the immediately preceding latent variable. This is commonly known as the “autoregressive” or the “simplex” model. The authors did not evaluate this model against the data partly because without further restrictions this is an underidentified model. Such a model can be tested with CTA. One vanishing tetrad,  $\tau_{1342} = 0$ , is implied by the model, and the CTA test statistic is a chi-square estimate of 1.12 with 1 *df* and a *p*-value of

0.29. The CTA results suggest a good fit for the four-wave path model. We are encouraged to explore this model with the conventional ML method by constraining  $\beta_{21} = \beta_{32} = \beta_{43}$  and  $\text{VAR}(\delta_1) = \text{VAR}(\delta_2) = \text{VAR}(\delta_3) = \text{VAR}(\delta_4)$ . The model with these equality constraints has an excellent fit with a chi-square of 2.93 with 4 *df* and a *p*-value of 0.57. This example shows that multiwave single indicator panel models can be tested with CTA procedures even when the model is underidentified.

Dimensionality tests are also possible with CTA. Suppose we wish to test whether intelligence is a stable latent variable that influences all four tests. Figure 6(b) contains the path diagram for this alternative model. Compared to Figure 6(a), this model assumes a one latent-variable solution rather than a four latent-variable one. The model in Figure 6(b) implies two independent vanishing tetrads. The CTA test statistic is 5.42 with 2 *df* and a *p*-value of 0.07. The vanishing tetrad for the four-wave simplex model shown in Figure 6(a) is nested in those implied in the one latent-variable model. The chi-square difference between these two models is 4.30 with 1 *df*. The *p*-value is less than 0.05, which suggests that the four-wave model in Figure 6(a) is preferable. Thus this example illustrates a tetrad test for dimensionality.

## 7. CONCLUSIONS

Confirmatory Tetrad Analysis holds promise as a model testing procedure in SEM. At a minimum, it provides a check on the LR test results. When both test statistics agree, it increases our confidence in a model's match to the data. Disagreements suggest potential specification errors or differences in the power of the tests. In addition, CTA applies in some situations where the LR test statistic is inapplicable or is more complicated to apply. We gave examples of models that were not nested for LR tests but were nested in their vanishing tetrads. Thus we can compare and test some models that have long been considered nonnested. Furthermore, the fit of some underidentified models can be assessed with the CTA test statistic. This could help researchers to determine whether it is worth looking for further restrictions that would help to identify a model. Finally, the test statistic we have used could also be helpful in ETA when comparing alternative models that have nested vanishing tetrads.

The characteristics of CTA may be made clearer by contrasting it with the more common maximum likelihood (ML) and weighted least squares (WLS) approaches to SEM. In ML/WLS approaches, the null hypothesis is  $H_o: \Sigma = \Sigma(\theta)$ , where  $\Sigma$  is the population covariance matrix of the observed variables and  $\Sigma(\theta)$  is the model implied covariance matrix with  $\theta$  the vector of free parameters in the model. We have a test statistic that has an asymptotic chi-square distribution when  $H_o$  is valid. In CTA the null hypothesis is  $H_o: \tau = \mathbf{0}$  where  $\tau$  is the vector of vanishing tetrads implied by the model. Here, too, we have a test statistic that has an asymptotic chi-square distribution when  $H_o$  is valid. In both cases the chi-square distribution is a large sample result; the small sample properties require further study. This suggests the need for Monte Carlo simulation experiments to explore the behavior of the test statistics in commonly used sample sizes.

Nesting of models in the ML/WLS approach occurs when the parameters of one model are a restricted version of the parameters in another model. Nesting in CTA exists when the vanishing tetrads of one model are a restricted version, typically a subset, of the vanishing tetrads of another model. As we illustrated here, a set of models can be nested in their vanishing tetrads but not nested in their parameters, and this allows a test of some models that are nonnested in their structural parameters.

With ML/WLS methods, it is possible that multiple models have identical values for the implied covariance matrix and for the test statistic. The equivalent models are indistinguishable in terms of their overall fit to the data (Jöreskog and Sörbom 1989, pp. 221–24). Similarly, we can have the same vanishing tetrads implied by multiple models. These “tetrad equivalent” models are indistinguishable in fit using our test statistic. Thus with both the ML/WLS and CTA approaches we should not confuse a favorable test statistic with proof of the validity of a model since other models can have a fit as good as or better than the ones tested.

The idea of tetrad equivalent models can help explain why it is possible for models to have tetrads that are nested but parameters that are not. A given set of vanishing tetrads can be implied by more than one model. The same is true for a second set of vanishing tetrads that is nested in the first. A test statistic for the nested tetrads provides a test of the relative fit of all models that imply the one set of vanishing tetrads to all models that imply the other set. Some of the models in

the two sets may be nested in their parameters, but, as we illustrated, this need not be true. And this leads to situations where we can compare the fit of models not nested in their parameters.

An important difference in methodologies is that ML/WLS is a structural parameter estimator, while CTA tests only model fit and does not estimate structural parameters.<sup>10</sup> For this reason, CTA clearly is a complement, not a replacement, for the traditional procedures. Largely because it is not a structural parameter estimator, CTA does not require iterative methods as do the ML/WLS methods.

We close by pointing out that CTA is in the original spirit of Spearman's work of determining the vanishing tetrads implied by a model and assessing whether they hold. It also is consistent with the early work on SEM that attempted to test models by examining the implied vanishing tetrads. This paper furthers the work of Spearman and others by providing a general simultaneous test of vanishing tetrads to evaluate models, by giving new rules for determining the vanishing tetrads implied by a model and eliminating the redundant ones, and by providing a method to estimate the power of the vanishing tetrad test.

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<sup>10</sup>Of course, the vanishing tetrads,  $\tau$ , are parameters that are estimated with CTA, but by structural parameters we are referring to the coefficients, variances, and covariances parameters that are part of the model structure.

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